

A Curious Paradox

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Consider two positive integers x and y , one of which is twice as great as the other. We are not told whether it is x or y that is the greater of the two. I will now prove the following two obviously incompatible propositions.

Proposition 1. *The excess of x over y , if x is greater than y , is greater than the excess of y over x , if y is greater than x .*

Proposition 2. *The two amounts are really the same (i.e., the excess of x over y , if x is greater than y , is equal to the excess of y over x , if y is greater than x).*

Proof of Proposition 1. Suppose x is greater than y . Then $x = 2y$, hence the excess of x over y is then y . Thus the excess of x over y , if x is greater than y , is y . Now, suppose y is greater than x . Then $x = \frac{1}{2}y$, hence the excess of y over x is then $y - \frac{1}{2}y = \frac{1}{2}y$. Thus the excess of y over x , if y is greater than x , is $\frac{1}{2}y$. Since y is greater than $\frac{1}{2}y$, this proves that the excess of x over y , if x is greater than y , is greater than the excess of y over x , if y is greater than x . Thus Proposition 1 is established. \square

Proof of Proposition 2. Let d be the difference between x and y — or what is the same thing, the lesser of the two. Then obviously the excess of x over y , if x is greater than y , is d , and the excess of y over x , if y is greater than x , is again d . Since $d = d$, Proposition 2 is established! \square

Now, Propositions 1 and 2 can't both be true! Which of the two propositions do you actually believe?

Most people seem to opt for Proposition 2. But look, suppose y , say, is 100. Then the excess of x over y , if x is greater than y , is certainly 100, and the excess of y over x , if y is greater than x , is certainly 50 (since x is then 50). And isn't 100 surely greater than 50?