# Some Diophantine Recreations

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There are a number of recreational problems that lead to equations to be solved. Sometimes the difficulty is in setting up the equations, with the solution then being easy. Other times the equations are straightforward, but the solution requires some ingenuity, such as exploiting the symmetry of the problem. In a third class of problems, the equations and their solution are relatively straightforward, but there is an additional diophantine requirement that the data and/or the answers should be integral. In this case, it is not always straightforward to determine when the problem has solutions or to find them all or to determine the number of solutions. I discuss three examples of such diophantine recreations. The first is a simple age problem, done to illustrate the ideas. The second is the Ass and Mule Problem, for which I have just found a reasonably simple condition for integer data to produce an integer solution. The third is the problem of Selling Different Amounts at the Same Price Yielding the Same, for which I give an algorithm for finding all solutions and a new simple formula for the number of solutions.

## A Simple Age Problem

Age problems have been popular since at least *The Greek Anthology* [16], compiled around A.D. 500 (although the date of this is uncertain, and the material goes back several centuries). The simplest are just problems of the 'aha' or 'heap' type, leading to one equation in one unknown. In *The Greek Anthology,* the problem of Diophantus' age gives us

$$\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x\,,$$

i.e., $75x/84 + 9 = x$ or $9x/84 = 9$ or $x = 84$. See Tropfke [20] for this and related problems.

In the late nineteenth century, complicated versions like "How Old Is Ann?" appeared, where the problem was in understanding the phrasing of the question, which contained statements such as "Mary is twice as old as Ann was when Mary was as old as Ann is now." (See Loyd [14])

Between these are a number of types of problem. The type that I want to examine is illustrated by the following, which is the earliest I have found, reportedly from *The American Tutor's Assistant,* 1791.[1]

When first the marriage knot was ty'd
Between my wife and me,
My age was to that of my bride
As three times three to three
But now when ten and half ten years,
We man and wife have been,
Her age to mine exactly bears,
As eight is to sixteen;
Now tell, I pray, from what I've said,
What were our ages when we wed?

A more typical, more comprehensible, albeit less poetic, version, but with the same numbers, appears at about the same time in Bonnycastle, 1824 [5].

> A person, at the time he was married, was 3 times as old as his wife; but after they had lived together 15 years, he was only twice as old; what were their ages on their wedding day?

We can state the general form of this problem as follows.

Problem $(a, b, c)$; $X$ is now $a$ times as old as $Y$; after $b$ years, $X$ is $c$ times as old as $Y$.

Thus the problem in *The American Tutor's Assistant* and Bonnycastle is Problem (3, 15, 2). It is easily seen that the problem leads to the equations

$$X = aY\,; \qquad X + b = c(Y + b)\,, \tag{1}$$

where $X$ and $Y$ denote the ages of $X$ and $Y$. The solution of these is easily found to be

$$Y = \frac{b(c-1)}{a-c}\,; \qquad X = aY = \frac{ab(c-1)}{a-c}\,, \tag{2}$$

assuming $a \neq c$.

Now we give a diophantine aspect to our problem by asking the question: *If $a$, $b$, and $c$ are integers, when are $X$ and $Y$ integers?* For example,

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[1] I haven't actually seen the book; the problem is quoted in Bunt, et al. [6].

Problem (4, 3, 2) does not have an integral solution. Since $X$ is integral if $Y$ is, we can give a pretty simple answer: $Y$ is integral if and only if

$$(3) \qquad a - c \quad \text{divides} \quad b(c-1)\,.$$

If we let $g$ be the greatest common divisor (GCD) of $a - c$ and $c - 1$, denoted as $g = (a - c, c - 1)$, then (3) holds if and only if

$$(4) \qquad \frac{a-c}{g} \quad \text{divides} \quad b\,.$$

This gives us the kind of condition that allows us to construct all solutions. We pick arbitrary integers $a$ and $c$, with $a > c > 1$, then let $b$ be any multiple of $(a - c)/g$.

This pretty much settles the problem, although one can consider permitting some or all of the values to be rational numbers. Nonetheless, one can still get unexpected results, as in the following problem:

My daughter Jessica is 16 and very conscious of her age. Our neighbour Helen is just 8, and I teased Jessica by saying "Seven years ago, you were 9 times as old as Helen; six years ago, you were 5 times her age; four years ago, you were 3 times her age; and now you are only twice her age. If you are not careful, soon you'll be the same age!"

Jessica seemed a bit worried, and went off muttering. I saw her doing a lot of scribbling.

The next day, she said to me, "Dad, that's just the limit! By the way, did you ever consider when I would be half as old as Helen?" Now it was my turn to be worried, and I began muttering — "That can't be, you're always older than Helen."

"Don't be so positive," said Jessica, as she stomped off to school.

Can you help me out?

* * *

It is also possible to find a situation in which one person's age is an integral multiple of another's during six consecutive years.

## Ass and Mule Problems

The classical ass and mule problem has the two animals carrying sacks. The mule says to the ass: "If you give me one of your sacks, I will have as many as you." The ass responds: "If you give me one of your sacks, I will have twice as many as you." How many sacks did they each have?

The general version of the problem for two individuals can be denoted $(a, b; c, d)$ for the situation where the first says: "If I had $a$ from you, I'd have $b$ times you," and the second responds: "And if I had $c$ from you, I'd have $d$ times you." Many versions of this problem occur, but it is traditional for the parameters $a, b, c, d$ and the solutions, say $x, y$, to be integers. The general solution of the problem gives somewhat complicated expressions for $x$ and $y$. Hence a natural question is to consider the diophantine question *Which integer problems have integer solutions?* In the early 1990s [1], I said I knew of no way to decide this that was simpler than actually seeing if the solutions were integers. Here I present a reasonably simple criterion that allows one to generate all the integer problems with integer solutions. I am indebted to S. Parameswaran for the letter that inspired this investigation.

The general problem $(a, b; c, d)$ leads to the system of equations

$$x + a = b(y - a)\,; \qquad y + c = d(x - c)\,. \tag{1}$$

The solutions are readily computed to be

$$x = c + \frac{(b+1)(a+c)}{bd-1}\,; \qquad y = a + \frac{(d+1)(a+c)}{bd-1}\,. \tag{2}$$

Thus, $x$ and $y$ are integers if and only if the second terms in (2) are integers. One can see from (2), and it is obvious from (1), that $x$ is an integer if and only if $y$ is an integer, so we only need to check one of the second terms in (2). This still doesn't give us a very simple or satisfactory criterion, however. Looking for a more symmetric solution, I first noted that $x + y$ has a symmetric expression, and I thought that the integrality of this was equivalent to integrality of $x$ and $y$; unfortunately, this doesn't hold. Luckily, though, this false trail led me to the following simple result, which is similar to, but somewhat more complex than, the age problem.

The values of $x$ and $y$ are integers if and only if $bd - 1$ divides both $(b + 1)(a + c)$ and $(d + 1)(a + c)$, which is if and only if $bd - 1$ divides GCD$[(b+1)(a+c), (d+1)(a+c)] = (a+c)(b+1, d+1)$, where $(b+1, d+1)$ denotes the GCD of $b + 1$ and $d + 1$, as before.

Consider $g = (b+1, d+1)$. This $g$ also divides $(b+1)(d+1) - (b+1) - (d+1) = bd - 1$. Now the last statement of the previous paragraph can be divided by $g$ to give us that $x$ and $y$ are integers if and only if

$$\frac{bd-1}{(b+1, d+1)} \quad \text{divides} \quad a + c\,. \tag{3}$$

This seems to be as simple a criterion for integrality as one could expect. The criterion allows us to pick arbitrary $b$ and $d$, assuming $bd \neq 1$, and then determines which values of $a$ and $c$ give integral solutions. I find it particularly striking that $a$ and $c$ only enter via the sum $a + c$. I am also intrigued to see that any $b$ and any $d$ can be used, assuming $bd \neq 1$, which I would not have predicted.

The basic role of $b+1$ and $d+1$ in the above leads one to ask whether the problem can be recast in some way to make these the natural parameters, but doing so doesn't seem to make the result significantly clearer.

Although the problem traditionally has all integral parameters, this is not really essential; the admission of irrational values, however, loses all number-theoretic interest. One can deal with rational $a, c, x, y$ by scaling the problem so that one has an integer problem, so it seems most interesting to assume that we have integral $a$ and $c$ and we want integral $x$ and $y$. It is then quite feasible to consider rational $b$ and $d$. Setting $b = \beta/\alpha$, $d = \delta/\alpha$, where GCD $(a, \beta, \delta) = 1$, it is direct to show that $x$ and $y$ are integral if and only if

$$(4) \qquad \frac{\beta\delta - \alpha^2}{(\beta + \alpha, d + a)} \quad \text{divides} \quad \alpha(a + c).$$

Dr. Parameswaran interpreted the problem as though the second statement was also made by the first animal. Thus he considered the problem where equations (1) are replaced by

$$(6) \qquad x + a = b(y - a)\,; \qquad\qquad x + c = d(y - c)\,.$$

This is easily solved to obtain

$$(7) \qquad y = a + \frac{(d+1)(a-c)}{b-d}\,; \qquad\qquad x = -a + \frac{b(d+1)(a-c)}{b-d}\,.$$

The expression for $x$ is a bit complicated, but we see from (6) that $x$ is an integer if $y$ is an integer, though the converse may fail. From the first equation, we get a fairly direct way to check whether $y$ is integral, but by analogy with the argument above, we see that $y$ is integral if and only if $b-d$ divides $(d+1)(a-c)$. Now consider $g = (b-d, d+1)$. We easily see that $g = (b+1, d+1)$, so Parameswaran's version has integral solutions if and only if

$$(8) \qquad \frac{b-d}{(b+1, d+1)} \quad \text{divides} \quad a - c\,.$$

From this, one can easily generate all examples.

Thinking about Parameswaran's version led me to wonder if there was any case where both versions gave integral answers, possibly even the same answers. That is, suppose you can hear the animals but can't tell which one is speaking, so you don't know whether the statements are made by the same or different animals. Are there problems where the answers are integral, or even the same, in either case? Perhaps surprisingly, there are problems where the answers are indeed the same, and I leave it for the reader to discover them.

**History of the Ass and Mule Problem.** The earliest known version of this problem has an ass and a mule with parameters (1, 2; 1, 1) and is attributed to Euclid, from about 300 B.C. Heiberg's edition [10] gives the problem in Greek and Latin verses.

Diophantus studied the problem in general. He proposes "to find two numbers such that each after receiving from the other may bear to the remainder a given ratio." He gives (30, 2; 50, 3) as an example. He also covers similar problems with three and four persons.

Two versions, (10, 3; 10, 5) and (2, 2; 2, 4), appear in *The Greek Anthology* [16]. Alcuin gives an unusual variant of the problem in that he assumes the second person starts from the situation after the transfer mentioned by the first person takes place. That is, the second equation of (1) would be replaced by

$$y - a + c = d(x + a - c)\,.$$

This is our problem $(a, b; c - a, d)$.

By about the ninth century, the problem was a standard, not only in Europe but also in India and the Arab world, and it has remained a standard problem ever since.

The problem is readily generalised to more than two participants, but then there are two forms, depending on whether 'you' refers to all the others or just the next in cyclic sequence. I denote these as I$-(a, b; c, d; \ldots)$ for the case where 'you' means all the others and II$-(a, b; c, d; \ldots)$ for the case where 'you' means just the next person in a cyclic sequence. The earliest three-person version that I know of appears in India, about A.D. 850, in the work of Mahavira [15]; the problem is of type I. The earliest four-person version I have seen is Arabic, in al-Karkhi (aka al-Karaghi) from about 1010 [13], and the problem is of type II.

Tropfke [20] discusses the problem and cites some Chinese, Indian, Arabic, Byzantine, and Western sources. The examples cited in the Chiu Chang Suan Shu [7] state, e.g., "If I had half of what you have, I'd have 50." This is a different type of problem and belongs in Tropfke's previous section. The other Chinese reference is a work of about A.D. 485 that has only been translated into Russian and may well be similar to the earlier Chinese example, so that, surprisingly, the Ass and Mule problem may not be in any Chinese source.

Below I give a summary of versions that I have noted up through Fibonacci, who typically gives many versions, including some with five persons, an inconsistent example, and extended versions where a person says, e.g., "If you give me 7, I'd have 5 times you plus 1 more." This includes all the relevant sources cited by Tropfke except Abu al-Wafa (only available in Arabic) and "Abraham," who is elsewhere listed as probably being from the early fourteenth century and hence after Fibonacci, although the material is believed to be based on older Arabic sources.

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|---|---|---|
| (1, 2; 1, 1) | c. 300 B.C. | Euclid |
| (30, 2; 50, 3) | c. 250 | Diophantus |
| (10, 3; 10, 5) | c. 510 | *The Greek Anthology* |
| (2, 2; 2, 4) | c. 510 | *The Greek Anthology* |
| (2, 1; 0, 2) | 9th century | Alcuin |
| I-(9, 2; 10, 3; 11,5) | c. 850 | Mahavira |
| I-(25, 3; 23, 5; 22, 7) | c. 850 | Mahavira |
| II-(1, 2; 2, 3; 3, 4; 4, 5) | c. 1010 | al-Karkhi |
| (100, 2; 10, 6) | 1150 | Bhaskara |
| (1, 1; 1, 10) | 1202? | Fibonacci |
| (7, 5; 5, 7) | 1202? | Fibonacci |
| I-(7, 5; 9, 6; 11, 7) | 1202? | Fibonacci, with the problem statement giving the 6 as a 7 |
| I-(7, 3; 8, 4; 9, 5; 11, 6) | 1202? | Fibonacci, noting that this system is inconsistent |
| I-(7, 2; 8, 3; 9, 4; 10, 5; 11, 6) | 1202? | Fibonacci, with the problem statement giving the 6 as a 7 |
| I-(7, 4; 9, 5; 11, 6) | 1202? | Fibonacci |
| I-(7, 3; 9, 4; 11, 5) | 1202? | Fibonacci, done two ways |

## Selling Different Amounts at the Same Price Yielding the Same

Although this problem is quite old, there seems to be no common name for it, hence the above title, which is rather a mouthful. The problem is treated in one form by Mahavira [15], Sridhara [18], and Bhaskara [4], but this form is less clearly expressed and has infinitely many solutions, so I will first describe the later form which is first found in Fibonacci [11]. I quote a version that appears as No. 50 in the first printed English riddle collection "The Demaundes Joyous," printed by Wynken de Worde in 1511 [9].

> A man had three daughters of three ages; which daughters he delivered, to sell, certain apples. And he took to the eldest daughter fifty apples, and to the second thirty apples, and to the youngest ten apples; and all these three sold in like many for a penny, and brought home in like such money. Now how many sold each of them for a penny?

'In like many' means 'the same number' and 'in like much money' means 'the same sum of money.' Anyone meeting this problem for the first time soon realises there must be some trick to it. Fibonacci has no truck with trick questions and gives the necessary information — the sellers sell part of their stock at one price and then the remainders at another price. This may happen by the sellers going to different markets, or simply changing prices after lunch, or lowering prices late in the day in an attempt to sell their remainders, or raising them late in the day when few items are left and new buyers arrive. Although the objects are usually eggs, or sometimes fruit, which normally are all of the same value, Tartaglia [19] justifies the different prices by having large and small pearls. Even with this trick exposed, it can take a fair amount of time to work out a solution, and it is not immediately obvious how to find or count all the solutions.

Let us denote the problem with numbers $c_1, c_2, \ldots$ by $(c_1, c_2, \ldots)$, so that the above is (10, 30, 50). This is historically by far the most common version. Fibonacci only gives two-person versions: (10, 30); (12, 32); (12, 33). Fibonacci is one of the few to give more than a single solution. He gives a fairly general rule for generating solutions [12], which would produce the positive solutions with the smaller price equal to 1, and he gives five solutions for his first version — but there are 55, of which 36 are positive. Fibonacci goes on to consider solutions with certain properties — e.g., if the prices are also given, is there a solution; find all solutions with one price fixed; find solutions where the amounts received are in some ratio, such as the first receives twice the second.

The question of finding all solutions for a three-person problem seems to have first been tackled in the 1600s. According to Glaisher [12], the 1612 and 1624 editions of Bachet [3][2] give a fairly general rule for this case and apply it to (20, 30, 40). In 1874, Labosne revised the material, dropping Bachet's rule and replacing it with some rather vague algebra; he added several more versions as well, for which he often gives fractional answers, distinctly against the spirit of the problem. The 1708 English edition of Ozanam [17], which is essentially the same as the first edition of 1694, considers (10, 25, 30) and gives two solutions. The extensively revised and enlarged edition of 1725 (or 1723) outlines a fairly general method and correctly says there are ten solutions, contrary to De Lagny's statement that there are six solutions. Neither Glaisher nor I have seen the relevant work of De Lagny, but we both conjecture that he was counting the positive solutions, of which there are indeed six.

During the 1980s, I tried several times to find an easy way to generate the solutions. I have now arrived at a reasonably simple approach that turns

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[2] I have not seen these editions.

out to be essentially a generalisation and extension of Ozanam's method. Glaisher's paper contains most of the ideas involved, but he is so verbose and gives so many pages of tabular examples, special cases, and generalizations that it is often difficult to see where he is going — it is not until the 51st page that he gives the solutions for the problem (10, 30, 50), which he starts considering on the first page. He also assumes that the $c_i$ are an arithmetic progression, but it is not until late in the paper that one sees that he solves versions that are not in arithmetic progression by the simple expedient of inserting extra values. By (4) below, this doesn't change the problem, but I prefer to deal with the original values. Ayyangar [2] gives a much simplified general method for the Indian version of the problem and says many of his solutions are not given by Glaisher's method on page 19 — although I can't tell if Glaisher intends this to be a complete solution. Ayyangar's method is quite similar to my method, so I later sketch his method in my notation and fill in a gap.

Most previous methods yielded solutions involving some choices, subject to some conditions, but did not thoroughly check that all solutions are obtained. No one else has noticed that the number of solutions in the Western version can be readily computed. Indeed, whenever Glaisher gives the number of solutions, he displays or indicates all of them.

Consider the problem $(c_1, c_2, \ldots, c_n)$, i.e., the $i^{\text{th}}$ person has $c_i$ items to sell. We can reorder them so that $c_1 < c_2 < \cdots < c_n$, and we assume $n > 1$ to avoid triviality. Let $a_i$ be the number of items that the $i^{\text{th}}$ person sells at the higher price $A$, and let $b_i$ be the number of items that the $i^{\text{th}}$ person sells at the lower price $B$. Then we have the following:

$$a_i + b_i = c_i\,; \tag{1}$$

$$Aa_i + Bb_i = C\,, \tag{2}$$

where $C$ is the constant amount received. We have $2n + 3$ unknowns connected by these $2n$ equations. We thus expect a three-dimensional solution set. Normally the objects being sold are indivisible, so we want the numbers $a_i, b_i, c_i$ to be integers.

Because of the symmetry between $A$ and $B$, we can assume and have assumed $A > B$. This makes $a_1 > a_2 > \cdots > a_n$ and $b_1 < b_2 < \cdots < b_n$. (One can reverse $A$ and $B$ in order to make the $a_i$s increase, but the amounts sold at the higher price are the smaller numbers and hence are easier to deal with.)

From $Aa_1 + Bb_1 = Aa_2 + Bb_2$, we get $A(a_1 - a_2) = B(b_2 - b_1)$. Hence $A/B$ is a rational number, and we can assume that $A$ and $B$ are integers with no common factor, i.e., that $(A, B) = \text{GCD}(A, B) = 1$. This is equivalent to scaling the prices in equations (2) and reduces the dimensionality of the solution set to two. This is quite reasonable, as solutions that differ only in price scale are really the same. Solutions in the literature, however, are often given with fractional values, in particular $B = 1/\beta$ for some integer $\beta$ — e.g., the most common solution of the (10, 30, 50) problem has $A = 3, B = 1/7$ — and the usual solutions have each person selling as many groups of $\beta$ as possible. In the present approach, these prices are considered the same as $A = 21, B = 1$, and they are considered to give the same sets of solutions. Glaisher considers $A = 3$, $B = 1/7$ with the condition that $C$ be integral, which restricts the solutions to a subset of those for the case $A = 21, B = 1$. In some versions of the problem, the yield $C$ is specified. This is also equivalent to scaling the prices, and the solution set is again two-dimensional, unless one imposes some further condition such as $B$ being the reciprocal of an integer.

Now use equation (1) to eliminate $b_i$ from (2), giving us

$$(A - B)a_i = C - Bc_i\,. \tag{3}$$

Subtracting each of these from the case $i = 1$ leaves us $n - 1$ equations:

$$(A - B)(a_1 - a_i) = B(c_i - c_1)\,. \tag{4}$$

Since $(A, B) = 1$ implies $(A - B, B) = 1$, it follows that (4) has an integral solution for $a_1 - a_i$ if and only if $A - B$ divides $c_i - c_1$ and this holds for each $i$ if and only if

$$A - B \quad \text{divides} \quad \mathrm{GCD}(c_i - c_1)\,. \tag{5}$$

It is now convenient to adopt the following:

$$\alpha_i = a_1 - a_i\,; \qquad \gamma_i = c_i - c_1\,; \qquad \Gamma = \mathrm{GCD}(\gamma_i)\,.$$

Thus (5) can be rewritten as $A-B$ divides $\Gamma$. This tells us that $d(A-B) = \Gamma$ for some $d$, and (4) gives us $a_i = Bd\gamma_i/\Gamma$, so that

$$\alpha_i \quad \text{is a multiple of} \quad \gamma_i/\Gamma\,. \tag{6}$$

Now we will take $\alpha_2$ as the first basic parameter of our solution, and we note that $1 \leqslant \alpha_2 \leqslant c_1$. From (6), we must have $\gamma_2/\Gamma$ dividing $\alpha_2$, and I claim that any such $\alpha_2$ generates an integer solution of our problem. From the case $i = 2$ of (4), we have $(A - B)\alpha_2 = B\gamma_2$. Let $\gamma = (\alpha_2, \gamma_2)$, so the previous equation becomes $(A - B)\alpha_2/\gamma = B\gamma_2/\gamma$, where $(A - B, B) = (\alpha_2/\gamma, \gamma_2/\gamma) = 1$. Hence $A - B = \gamma_2/\gamma$, $B = \alpha_2/\gamma$. We also get $A = (\alpha_2 + \gamma_2)/\gamma$, $A/B = (\alpha_2 + \gamma_2)/\alpha_2 = 1 + \gamma_2/\alpha_2$. From just before (6), we have $\Gamma\alpha_2 = Bd\gamma_2$, so $\Gamma\alpha_2/ = Bd\gamma_2/\gamma$. Since $(\alpha_2/\gamma, \gamma_2/\gamma) = 1$, this implies that $\gamma_2/\gamma$ divides $\Gamma$, i.e., $A - B$ divides $\Gamma$, which is the condition (5) for all the equations (4) to have integral solutions.

As an example, consider the problem discussed in Ozanam, namely (10, 25, 30). Then $\gamma_2 = 15$, $\gamma_3 = 20$, $\Gamma = 5$, and we choose $\alpha_2$ as a multiple of $\gamma_2/\Gamma = 3$.

For $\alpha_2 = 3$, we have $\gamma = (3, 15) = 3$, $A - B = 5$, $B = 1$, $A = 6$, and there are solutions with $\alpha_i = \gamma_i/5$, i.e., $a_1 - a_2 = 3$, $a_1 - a_3 = 4$, hence $a_2 - a_3 = 1$. We can now let $a_1$ vary as the second basic parameter of our solution and we can let $a_1 = 10$, 9, 8, 7, 6, 5, 4, giving us seven solutions.

For $\alpha_2 = 6$, we have $\gamma = (6, 15) = 3$, $A - B = 5$, $B = 2$, $A = 7$, and there are solutions with $\alpha_i = 2\gamma_i/5$, i.e., $a_1 - a_2 = 6$, $a_1 - a_3 = 8$, hence $a_2 - a_3 = 2$. We can now let $a_1 = 10$, 9, 8, giving us three solutions.

For $\alpha_2 = 9$, we get no solutions.

Thus Ozanam is correct, but four of these solutions have either $b_1 = 0$ or $a_3 = 0$, so there are just six positive solutions.

**Counting the Solutions.** We get solutions in the above section for each $\alpha_2$ that is a multiple of $\gamma_2/\Gamma$ and in the interval $1 \leqslant \alpha_2 \leqslant c_1$. For such an $\alpha_2$, we can let $a_1 = c_1, c_1 - 1, \ldots,$ until $a_n$ becomes zero. From the case $i = n$ of (4), we have $(A - B)(a_1 - a_n) = B\gamma_n$, so that $a_n = 0$ when

$$a_1 = \frac{B\gamma_n}{A - B} = \frac{\alpha_2\gamma_n}{\gamma_2}\,. \tag{7}$$

(This expression is an integer, since $A - B$ divides $\gamma_n$.) Hence, for a given $\alpha_2$, there are

$$c_1 - \frac{\alpha_2\gamma_n}{\gamma_2} + 1 \qquad \text{solutions.} \tag{8}$$

In the above example, $\alpha_2 = 3$ gives $10 - 3 \times 20/15 + 1 = 7$ solutions; $\alpha_2 = 6$ gives $10 - 6 \times 20/15 + 1 = 3$, and $\alpha_2 = 9$ gives a negative value, indicating no solutions.

We can now determine the possible values of $\alpha_2$. From the case $i = n$ of (4), we have

$$\alpha_n = \frac{B\gamma_n}{A - B} = \frac{\alpha_2\gamma_n}{\gamma_2}\,. \tag{9}$$

Now the largest value $\alpha_n$ can have is $c_1$, so we have

$$1 \leqslant \alpha_2 \leqslant \frac{\gamma_2 c_1}{\gamma_n}\,. \tag{10}$$

In our example, we find $1 \leqslant \alpha_2 \leqslant 7.5$.

Recalling that $\alpha_2$ is a multiple of $\gamma_2/\Gamma$, set

$$\alpha_2 = \frac{k\gamma_2}{\Gamma}\,. \tag{11}$$

Then (10) becomes

$$1 \leqslant k \leqslant \frac{\Gamma c_1}{\gamma_n}. \tag{12}$$

The upper bound may not be an integer, so we let

$$K = \left[\frac{\Gamma c_1}{\gamma - n}\right], \qquad \text{where [ ] is the greatest integer function.} \tag{13}$$

Substituting (11) into (8) and summing gives us the number of solutions,

$$N = \sum_{k=1}^{K}(c_1 - \frac{k\gamma_n}{\Gamma} + 1) = K(c_1 + 1) - \frac{\gamma_n K(K+1)}{2\Gamma}. \tag{14}$$

Again, in our example, we have $K = 2$ and

$$N = 2 \times 11 - 20 \times 2 \times 3/2 \times 5 = 10.$$

Most of the earlier writers excluded zero values and only considered positive solutions. We can count the number $N_+$ of positive solutions by finding the number of solutions with a zero value. These occur when $b_1 = 0$, i.e., $a_1 = c_1$, and when $a_n = 0$, i.e., $a_1 = \alpha_2\gamma_n/\gamma_2$ by (7). Hence we normally get $2K$ situations with a zero value, unless it happens that $c_1 = \alpha_2\gamma_n/\gamma_2$, when two zero values occur in the same solution. Rearranging, we find that this occurs when $k = \Gamma c_1/\gamma_n$, i.e., when the upper bound for $k$ given in (13) is an exact integer. In that case, we have $2K - 1$ solutions with zero values. Subtracting $2K$ or $2K - 1$ from $N$ yields $N_+$.

Looking again at our example, $N_+ = 10 - 2 \times 2 = 6$, perhaps as intended by De Lagny.

Most of the examples in the literature have the $c_i$ in arithmetic progression (AP). A little thought shows that then $\Gamma$ is the common difference of the progression, so we can write $c_i = c_1 + (i-1)\Gamma$. Then we have $\gamma_n = (n-1)\Gamma$; $\gamma_n/\Gamma = n - 1$:

$$K = \left[\frac{c_1}{n-1}\right]; \qquad N = K(c_1 + 1) - \frac{(n-1)K(K+1)}{2}. \tag{15}$$

A number of early cases have $n = 2$, which is trivially an AP. We then have $K = c_1$; $N = c_1(c_1 + 1)/2$; $N_+ = (c_1 - 2)(c_1 - 1)/2$.

The majority of examples are APs with $n = 3$. All but one of these have $c_1$ even, and then $K = c_1/2$; $N = c_1^2/4$; $N_+ = (c_1 - 2)^2/4$. If $c_1$ is odd, then we get $K = (c_1 - 1)/2$; $N = (c_1^2 - 1)/4$; $N_+ = [(c_1 - 2)^2 - 1]/4$.

A few longer APs occur — I have seen examples with $n = 5, 7, 9$. A few cases occur that are not APs (see below).

Tartaglia's problem 139 is the one with two sizes of pearls and is (10, 20, … , 90). From (15), we have $K = 1$; $N = 3$; $N_+ = 1$. This led me to ask when a problem admits only one solution or only one positive solution. Glaisher [12] also studied this problem and was intrigued by this observation, noting the analogous cases with 10 and 11 people.

In general, $N = 1$ implies that the only solution occurs with $k = 1$ and has $a_1 = c_1$ and $a_n = 0$, which gives us $k = \Gamma c_1/\gamma_n$, hence $\gamma_n = \Gamma c_1$ or $c_n = (\Gamma + 1)c_1$. For an AP, this is if and only if $c_1 = n - 1$. For example, (10, 20, ... , 110) has $N = 1$.

Similarly, $N_+ = 1$ implies that the unique positive solution occurs with $k = 1$ and has $a_1 = c_1 - 1$ and $a_n = 1$. A little manipulation shows that this works if and only if

$$c_n = (\Gamma + 1)c_1 - 2\Gamma \,. \tag{16}$$

For an AP, this turns out to be if and only if $c_1 = n + 1$, as in Tartaglia's example.

Initially I thought that $N_+ = 1$ would imply that $N = 3$, but when $n = 2$, we find $K = 3$ and $N = 6$, while when $n = 3$, we have $K = 2$ and $N = 4$. We will get $N = 3$ only if $K = 1$, i.e., $\Gamma c_1/\gamma_n < 2$. Combining this with (16) shows that this implies $n > 3$, so the above gives all cases where $N_+ = 1$ and $N > 3$.

Below I tabulate all the different versions that I have noted, along with the first known (to me) dates and sources, the numbers of solutions, given as $(N, N_+)$, and the number of solutions given in the source. Tropfke [20] and Glaisher [12] cite a number of sources that I have not seen.

**The Indian Version** As mentioned earlier, the Indian version is somewhat different and gives infinitely many solutions. To illustrate, consider the following problem from Bhaskara, used as an example in Ayyangar.

> Example instanced by ancient authors: a stanza and a half. Three traders, having six, eight, and a hundred, for their capitals respectively, bought leaves of betle [or fruit] at an uniform rate; and resold [a part] so: and disposed of the remainder at one for five panas; and thus became equally rich. What was [the rate of] their purchase? and what was [that of] their sale?

This requires some explanation. Here the numbers $c_i$ are not the numbers of objects, but the capitals (in panas) of each trader. It is assumed that you can buy $D$ items per pana, so the the numbers of items will be $Dc_i$ and our equation (1) becomes

$$a_i + b_i = Dc_i \,. \tag{1'}$$

Further, we are specifically given the greater price $A = 5$, and it is understood that the lesser price corresponds to selling $\beta$ items per pana, i.e.,

| Version | Date | Source | # of Solutions | Solutions in Source |
|---|---|---|---|---|
| (10, 30) | 1202? | Fibonacci | (55, 36) | 5 given. |
| (12, 32) | 1202? | Fibonacci | (78, 55) | 6 given. |
| (12, 33) | 1202? | Fibonacci | (78, 55) | 1 given. |
| (10, 20) | c. 1300 | "Abraham" | (55, 36) | |
| (10, 30, 50) | 1300s | Munich codex 14684 | (25, 16) | 1 given. |
| (30, 56, 82) | 1489 | Widman | (225, 196) | 1 given. |
| (17, 68, 119, 170) | 1489 | Widman | (45, 35) | 1 given. |
| (305, 454, 603, 752, 901) | 1489 | Widman | (11,552; 11,400) | 1 given. |
| (10, 20, 30) | c. 1500 | Pacioli | (25, 16) | 1 given. |
| (8, 17, 26) | 1513 | Blasius | (16, 9) | 1 given. |
| (20, 40, 60) | 1515 | Tagliente | (100, 81) | 1 given. |
| (10, 50) | 1521? | Ghaligai | (55, 36) | 1 given. |
| (11, 33, 55) | 1556 | Tartaglia | (30, 20) | 1 given. |
| (16, 48, 80) | 1556 | Tartaglia | (64, 49) | 1 given. |
| (10, 20, . . ., 90) | 1556 | Tartaglia | (3, 1) | 1 given. |
| (20, 30, 40) | 1612 | Bachet | (100, 81) | 4 given. |
| (10, 25, 30) | 1725? | Ozanam from De Lagny | (10, 6) | all given. |
| (18, 40) | 1874 | Labosne | (171, 136) | 2 given, one with fractions. |
| (18, 40, 50) | 1874 | Labosne | (3, 1) | 1 given with fractions. |
| (10, 12, 15) | 1874 | Labosne | (7, 4) | none given. |
| (31, 32, 37) | 1874 | Labosne | (70, 60) | none given. |
| (27, 29, 33) | 1893 | Hoffmann | (117, 100) | 1 given. |
| (20, 30, . . . , 60) | 1905 | Dudeney | (45, 36) | 1 given. |
| (20, 40, . . ., 140) | 1924 | Glaisher | (27, 21) | 1 given. |

$B = 1/\beta$, where $\beta$ is an integer. Ayyangar's translation of the problem makes this more specific by saying they 'sold a part in lots and disposed of the remainder at one for five panas.' Thus (2) becomes

$$5a_i + \frac{b_i}{\beta} = C\,. \tag{2'}$$

We now have $2n + 3$ unknowns and $2n$ equations, so we again have a three-dimensional solution set. One can scale the set $D$, $a_i$, $b_i$, $C$ in some way to reduce the solution set to two dimensions, though it is not as easy to see how to do this as in the Western version. At this point, I must point out that some Indian versions give $c_i$ and/or $A$ as fractions! The Indian authors do not get very general solutions of these problems.

Sridhara finds a one-parameter solution set. Bhaskara makes the following comment: “This, which is instanced by ancient writers as an example of a solution resting on unconfirmed ground, has been by some means reduced to equation; and such a supposition introduced, as has brought out a result in an unrestricted case as in a restricted one. In the like suppositions, when the operation, owing to restriction, disappoints; the answer must by the intelligent be elicited by the exercise of ingenuity.” Amen!

For ease of expression, we let $\beta_i = b_i/\beta$, so our equations become

$$(1'') \qquad a_i + \beta\beta_i = Dc_i\,;$$

$$(2'') \qquad Aa_i + \beta_i = C\,.$$

Ayyangar gives a reasonable solution process for this problem, but since his notation is different, I will sketch the solution in the present notation, pointing out an improvement. Eliminating $\beta_i$, then subtracting to eliminate $C$ and using the same abbreviations as before, we get

$$(4') \qquad D\beta_i = (A\beta - 1)\,\alpha_i$$

and the condition for all the equations to have integral solutions is

$$(5') \qquad A\beta - 1 \quad \text{divides} \quad D\Gamma\,.$$

If we set

$$(5'') \qquad D\Gamma = k(A\beta - 1)\,,$$

then (4′) gives

$$(6') \qquad \alpha_i = \frac{k\gamma_i}{\Gamma}\,.$$

We need to determine which $k$ and $\beta$ will make (5″) hold, i.e.,

$$(5''') \qquad k(A\beta - 1) \equiv 0 \pmod{\Gamma}\,.$$

Let $\gamma = (k, \Gamma)$. Then (5‴) is equivalent to $A\beta - 1 \equiv 0 \pmod{\Gamma/\gamma}$ and this is solvable if and only if $(A, \Gamma/\gamma) = 1$. This gives us a rather peculiar condition. Write $\Gamma = rs$, where $r$ comprises all the prime powers in $\Gamma$ whose primes also occur in $A$. Then $\Gamma/\gamma$ is relatively prime to $A$ if and only if $\Gamma/\gamma$ divides $s$, which is if and only if $r$ divides $\gamma = (k, \Gamma)$, which is if and only if $r$ divides $k$. (Ayyangar notes only that $(A, \Gamma)$ must divide $k$, and $(A, \Gamma)$ can be a proper divisor of $r$.)

Both $k$ and $\beta$ can take on an infinite range of values, but in order for $\alpha_1 = k\gamma_n/\Gamma \leqslant Dc_1$, we must have

$$(7') \qquad \beta \geqslant \frac{c_n}{Ac_1}\,.$$

For given $k$ and $\beta$ satisfying ($5'''$) and ($7'$), there are only a finite number of solutions. Analysis similar to that done before shows that there are then $1 + [k(A\beta c_1 - c_n)/\Gamma\beta]$ solutions.

By the way, Bhaskara solves his example by taking $k = 2$, so $k = \Gamma$ and $\beta = 110$, so $D = 549$, giving total numbers $Dc_i$ of 3295; 4392; 54,900 and $a_i$ of 3294; 3292; 3200 and $C = 16{,}470$ [4].

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