An Update on Odd Neighbors and Odd Neighborhoods

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Some time ago (1986–88) this problem appeared in mathematical circles:

Problem 1: Can the unit squares of an $n \times n$ sheet of graph paper be labeled with 0's and 1's so that every neighborhood is odd?

A *neighborhood*, N_i, of square i, is the set of squares sharing an edge with square i and does include square i. A neighborhood, N_i is *odd* if there are an odd number of squares in N_i with labels "1".

Figure 1 shows a 4×4 example with a desired labeling.

1	1	0	1
1	1	1	0
0	1	1	1
1	0	1	1

Figure 1.

The first solution of Problem 1 was said to be quite difficult. Later, cellular automata ideas were used. However, if the problem is generalized to all graphs the proof is quite elementary.

Theorem 1. *Let G be a graph with no loops or multiple edges. Let A be the adjacency matrix of G. Then the vertices of G can be labeled with 0's and 1's so that every neighborhood is odd.*

Proof:[1] Consider the matrix equation $(A+I)x = 1$ where 1 is a column of 1's. We need to show that this equation has a solution for every adjacency matrix A. The linear algebra is, of course, over $\mathbf{Z}_2$, the integers modulo 2, since we are interested in only odd/even properties. Suppose y is a $1 \times n$ vector of 0's and 1's. Then

$$y(A+I) = 0 \iff yA = yI \iff Ay^T = yIy^T = yy^T.$$

[1]Due to the author's son, Bryan, while a graduate student.

But A is symmetric, so $yAy^T = 0$. Therefore, $yy^T = 0$ and $\text{rank}(A+I) = \text{rank}(A+I|1)$, so $(A+I)x = 1$ for every adjacency matrix A. Finally, for every graph G, the vertices of G can be labeled with 0's and 1's so that every neighborhood in G is odd.

If we think of the unit squares in Problem 1 as vertices of G and $v_i v_j$ is an edge in $G \iff v_i$ shares an edge with v_j, then Theorem 1 implies that the squares of the $n \times n$ square can be labeled with 0's and 1's so that each neighborhood is odd.

Now for some new questions (at least at the time that this paper was submitted).

The Developer's Dilemma

Suppose a real estate developer has an infinite (at least sufficiently long) strip of land m lots wide.

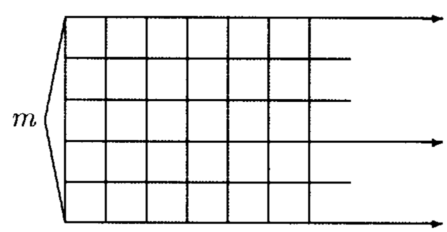

Figure 2. $m = 5$.

Further, we assume all lots in the first strip of m lots have been sold. It is quite possible that some of the buyers are odd, and some are not odd. Is there an n so that the developer can sell all the lots in the $m \times n$ rectangle and satisfy the politically correct criteria that all neighborhoods have an odd number of odd neighbors?

When this paper was originally written, the answer to the Developer's Dilemma was unknown. We will generalize, as before, to all graphs and prove a theorem. But this time the generalization does not yield a solution to the Developer's Dilemma. However, a proof that the Dilemma can be solved will also be presented.

Theorem 2. *Let G be a graph with vertices labeled with 0's and 1's in any way. If not all neighborhoods of G are odd, then G can be embedded in a graph G', with all neighborhoods in G' odd.*

It is rather interesting that only one vertex must be added to G in order to prove Theorem 2.

Proof. Let G' be the graph formed from G by adding a new vertex, w. Edges from w to vertices $v \in G$, with N_v even, are also added. Label w with a 1. Clearly all neighborhoods N_v with $v \neq g, v \in G'$ are now odd in G'. We need only to show N_w is odd in G'. Let D_k be the number of vertices labeled 1 and adjacent to vertex $k \in G$. N_k is even in $G \iff D_k$ is odd and N_k is odd in $G \iff D_k$ is even. But,

$$\sum_{\substack{k \in G \\ k \text{ labeled } 1}} D_k \text{ is even,}$$

since the sum counts the edges from vertices labeled 1 exactly twice. Hence, there are an even number of vertices $k \in G$ labeled 1 and N_k is even in G. These vertices now are all connected to w with label 1. So N_w is odd $\in G'$.

Theorem 3. *The Developer's Dilemma can be solved.*

Proof. Let the $m \times \infty$ strip of unit squares be horizontal. We call v_1, any 0–1 m-tuple, a "starter" for the $m \times n$ rectangle we seek. We wish to construct a proper labeling of an $m \times n$ rectangle so that all neighborhoods are odd and the vector v_1 is the "starter" column.

Perhaps the $m \times 1$ rectangle using the starter v_1 has the property that all neighborhoods are odd. Figure 3 shows that this phenomenon can happen.

1
0
0
1
0
0
1

(a)

a	x_1
b	x_2
c	x_3
d	x_4
e	x_5

(b)

Figure 3.

If not, all neighborhoods are odd, and we describe an algorithm that can be used to label the next column so that all N_{1k} are odd for v_{1k} and unit square in column 1. (See Figure 3(b).)

If $a+b$ is odd, let $x_1 = 0$, otherwise $x_1 = 1$.
If $a+b+c$ is odd, let $x_2 = 0$, otherwise $x_2 = 1$.
If $b+c+d$ is odd, let $x_3 = 0$, otherwise $x_3 = 1$.
If $c+d+e$ is odd, let $x_4 = 0$, otherwise $x_4 = 1$.
If $d+e$ is odd, let $x_5 = 0$, otherwise $x_5 = 1$.

Of course, m is not limited to the value 5. Now that every neighborhood for v_{1k} is odd, we can repeat the process for column 2. Each neighborhood will now have all but one entry labeled, and that label can be assigned so that the neighborhood is odd.

This process can be repeated as many times as we like, and each new column will be dependent on just the preceding two. We will have a solution to our problem if the $n+1$ column has all entries 0, i.e., N_{nk} is odd for square v_{ik} in column $n, i \leq n$. Since there are only a finite number of adjacent pairs of the m-tuple, either we get a column of 0's or there must be an adjacent pair that repeats in our process. Let $v_i v_{i+1}$ be the first pair that repeats. We can generate new columns moving from left to right or from right to left, and we generate a second $v_i v_{i+1}$ somewhere to the right of the first $v_i v_{i+1}$ (see Figure 4).

$$\begin{array}{ccccccc} v_1 & \cdots & v_i & v_{i+1} & \cdots & v_i & v_{i+1} \\ \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow \end{array}$$

Figure 4.

Using the second occurrence of $v_i v_{i+1}$, column v_{i-1} must repeat to the left of the first occurrence of $v_i v_{i+1}$.

$$v_1 \cdots v_{i-1} v_i v_{i+1} \cdots v_{i-1} v_i v_{i+1}$$

So $v_i v_{i+1}$ is not the first repeating pair. Contradiction! Hence there is a column of 0's and the proof is complete.

Figure 5 shows an example where $v_1 = \text{col}(00110)$, and $n = 7$.

0	1	0	1	0	0	0	0
0	0	1	0	0	1	1	0
1	1	0	1	0	1	1	0
1	1	0	0	1	0	0	0
0	0	0	1	0	1	0	0

Figure 5.

Perhaps the next question might be *For m fixed, is there an n such that for every starter v the $m \times n$ rectangle has all neighborhoods odd?, i.e., n is independent of v.*

For $m = 5$, to show that $n = 23$ is a nice exercise for the reader.