Self-Designing Tetraflexagons

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Flexagons are paper structures that can be manipulated to bring different surfaces into view. The four-sided ones discussed here are "flexed" by folding them in half along an axis, and then opening them up a different way to reveal a new face. Sometimes the opening is in a different direction (mountain or valley fold), sometimes along the other axis, and sometimes both. Although usually constructed from a strip of paper with attachment of the ends, flexagons seem more elegant when no glue or paste is required. Interesting designs of the faces can be found when the material is paper, or cardboard, with a different color on each side (as in standard origami paper). I call these "self-designing." What follows is the result of my explorations.

Although the text refers only to a square, the base for construction can be any rectangle. There are five different starting bases (paper shapes; see Figures 1–5). There are three different ways of folding the bases. Warning: Not all the results are interesting. But some are, my favorites being three: the first version of the cross plus-slit (for its puzzle difficulty); the first version of the square slash-slit (for the minimalist base and puzzle subtlety); and the third version of the square cross-slit (for its self-design). Note that the text assumes you are using two-colored paper, referred to, for simplicity, as black and white.

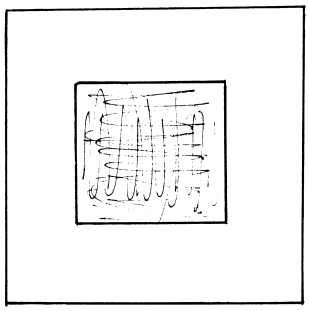

Figure 1. Square window.

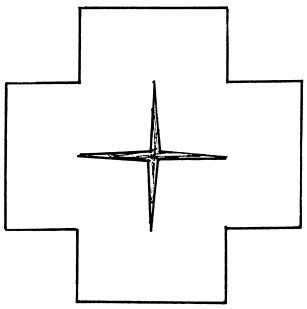

Figure 2. Cross plus-slit.

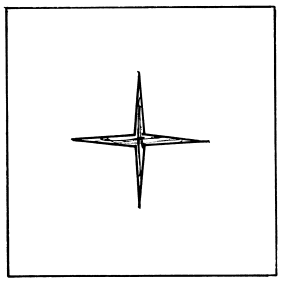

Figure 3. Square plus-slit.

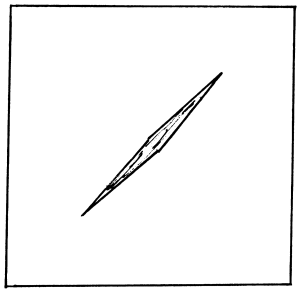

Figure 4. Square slash-slit.

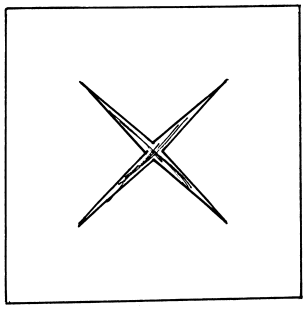

Figure 5. Square cross-slit.

Square Window

The first flexagon I saw that did not need glue was shown to me by Giuseppi Baggi years ago. It is a hexa-tetraflexagon made from a "window" — a square (of 16 squares) with the center (of 4 squares) removed (Figure 1). It has six faces that are easy to find.[1] I have recently used this window model for a routine about the riddle of the chicken and egg: "What's Missing Is What Comes First." This involves decorating the paper with markers, so is a separate manuscript.

It should be noted, however, that there are two ways of constructing this flexagon, both of which are discussed below.

Same Way Fold. As the dotted line indicates in Figure 6, valley fold the left edge to the center. The result is shown in Figure 7. As the dotted line indicates in Figure 7, valley fold the upper edge to the center. The result is

[1]Directions for constructing the hexa-tetraflexagon are found on pages 18–19 of Paul Jackson's *Flexagons,* B.O.S. Booklet No. 11, England, 1978.

shown in Figure 8. As the dotted line indicates in Figure 8, valley fold the right edge to the center. The result is shown in Figure 9.

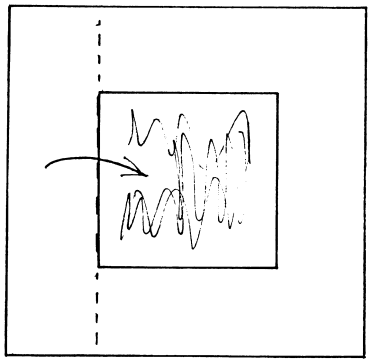

Figure 6.

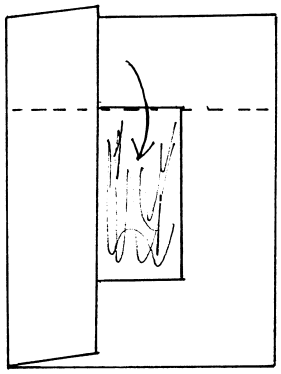

Figure 7.

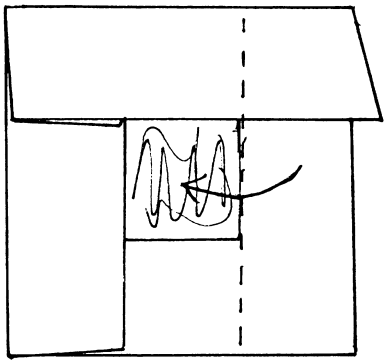

Figure 8.

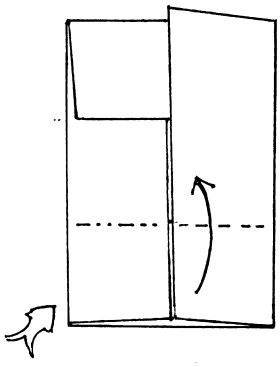

Figure 9.

The final fold is very tricky to communicate, but not at all tricky to do when you understand it. The goal is to make this last corner exactly the same as the other three. As you would expect, the bottom edge will be valley folded to the center. The right half of this lower portion falls directly on top of the portion above it. The left half of this lower portion goes underneath the portion above it. (So Figure 9 shows a valley fold on the right half and a mountain fold on the left half.) To make this actually happen, lift up the upper layer only of the left half of the lower portion, and then fold the entire lower portion to the center. The result is shown in Figure 10.

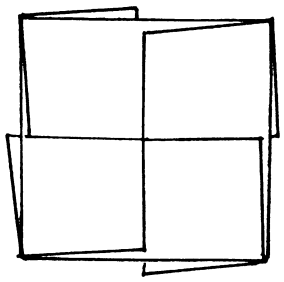

Figure 10.

Note that the model is entirely symmetrical. The four corners are identical, and both sides of the model are identical. (If this is not the case, you have made a mistake, probably on the last move.) This method amounts to folding the edges to the center, one after another (proceeding either clockwise or counterclockwise around the square), all of the folds being valley folds. This produces a Continuous Flex, moving in a straightforward manner from face 1 to 2 to 3 to 4 to 1, 5, 6, 1. The designs procured are merely three black faces and three white faces, as follows: black, black, white, white, black, black, white, white, black.

Alternating Way Fold. As Figure 11 indicates, mountain fold the left edge to the center. The result is shown in Figure 12. As the dotted line indicates in Figure 12, valley fold the upper edge to the center. The result is shown in Figure 13. As the dotted line indicates in Figure 13, mountain fold the right edge to the center. The result is shown in Figure 14.

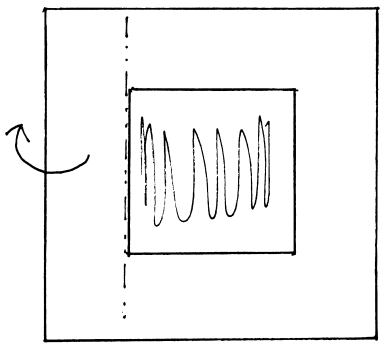

Figure 11.

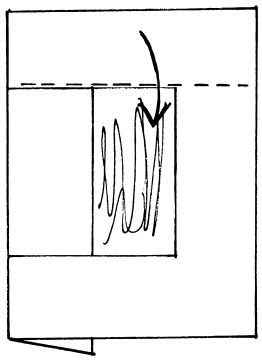

Figure 12.

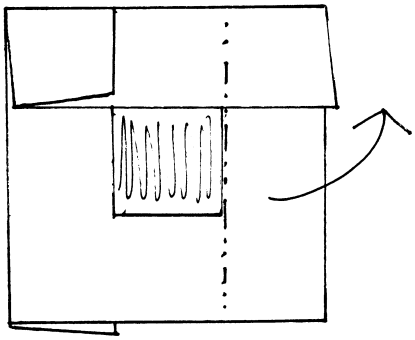

Figure 13.

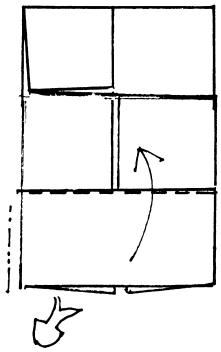

Figure 14.

Before you make the final valley fold of the bottom edge to the center, pull the bottom layer at the lower left corner away from the upper layer. Make the valley fold, and allow that bottom layer to go back to the bottom of the lower left corner. The result is shown in Figure 15. Note that opposite corners are identical, and both sides of the model are identical.

This noncontinuous Puzzle Flex can be flexed continuously through four faces only: 1 to 2 to 3 to 4 to 1, etc. Finding the other two faces is easy in this particular case, but backtracking is required. The four faces are identical: a checkerboard pattern of two black squares and two white squares. The other two faces are a solid black face and a solid white face.

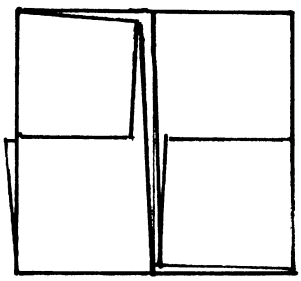

Figure 15.

Cross Plus-Slit

The window model inspired me to think of other ways to form a flexagon without having to attach the ends to each other. One result was a puzzle made from a "cross" — a square (of 16 squares) with the four corner squares removed, and two slits in the center that intersect in the shape of a plus sign (see Figure 2).

This base is manipulated in a third way, the 3-D Tricky Way. This puzzle is interesting for two reasons: It is tricky to construct, and, while four of the faces are easy to find, the other two faces are quite difficult to discover, involving changing the flat flexagon into a three-dimensional ring and back again. The four faces are continuous and identical, being black and white checkerboard patterns. The other two faces are solid black and solid white. (Note: The trick fold for constructing the flexagon can be done in a slightly different way that has the four faces not continuous.)

3-D Tricky Fold In order to follow the instructions and understand the diagrams, number the base from one to six, exactly as indicated in Figure 16. Orient them just as indicated. The numbers in parentheses are on the back side of the base. You will make two, very quick, moves. The first renders the base 3-D, and the second flattens it again.

Note the two arrows in Figure 16. They indicate that you are to make the base 3-D by pushing two opposite corners down and away from you. So reach underneath and hold the two free corners of the 1 cells, one corner in each hand. When you pull them down and away from each other, they are turned over so you see the 5 cells. The other cells come together to form two boxes open at the top. There is a 6 cell at the bottom of each box

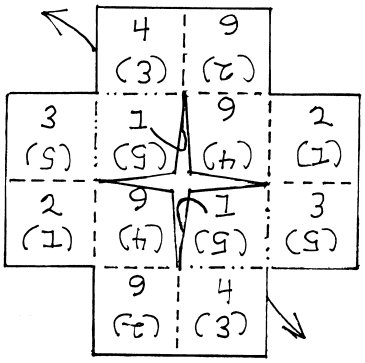

Figure 16.

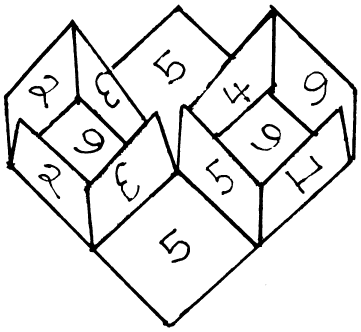

Figure 17.

(see Figure 17). Now the base should be flattened into a compact model with four cells showing on a side. Change your grip so you are holding the two inside corners of the 5 cells. (These are the corners opposite the ones you were holding.) Now push down on these corners, and at the same time, pull them away from each other. The 3-D boxes will collapse, the base forming a flat square of four cells. Once this happens, you should have four 1 cells facing you, and four 4 cells on the other side. The result is shown in Figure 18.

Sometimes, however, you will find that another number has appeared, a 2 instead of a 1, or a 3 instead of a 4. You can correct this easily by tucking the wrong cell out of sight, replacing it with the proper one. Check both sides. The model is completed.[2]

Flexing. You can find the faces numbered 1 to 4 by the usual flexing. Faces 5 and 6 are found by the following procedure. Begin with face 1 on the top and face 4 on the bottom. Mountain fold the model in half on the vertical axis, the left and right halves going back away from you. Do not open the model as you do when flexing. Rather, move the lower inside packet of squares (with 2 and 3 on the outside) to the left, and the upper inside packet of squares (with 2 and 3 on the outside also) to the right. Now open the model into a tube — a cube open at both ends. Collapse the tube in the opposite way. This creates a new arrangement. Flex it in the usual way to show face 5, then 3, and then make the tube move again

[2]Other directions for constructing this flexagon are on page 27 of Jackson's *Flexagons*, and in Martin Gardner's *Wheels, Life and Other Mathematical Amusements*, New York: W.H. Freeman & Co., 1983, pp. 64–68.

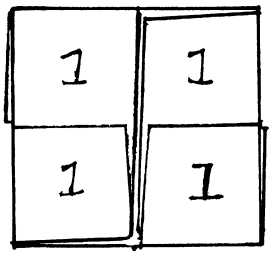

Figure 18.

to return to face 4 and then face 1. To find face 6, begin on face 4, with face 1 at the back, and repeat the moves just given.

Another version of this flexagon can be made by using the Same Way Fold. (Alternating Way gets you nowhere.) Four faces are easily found, but they are not continuous. Two faces can be found only by forming the rings, as mentioned above. The designs of the four faces are two solid black and two checkerboard. The other two faces are solid white.

Square Plus-Slit

About a year or so ago, I discovered a way to make a flexagon without removing any portion of the square. (The goal was inspired by my desire to make a flexagon from paper money with minimal damage.) It is made from a square with two slits intersecting the center in the shape of a plus sign (Figure 3).

For best results, use the Alternating Way fold. Flexing is continuous, and four faces can be shown: checkerboard, solid white, solid black, checkerboard. The Same Way Fold can be used also, but with poor results. Four faces can be shown, but they are not continuous. Indeed, both the third and fourth are derived from the first face. All faces are of the identical solid color. Using the 3-D Tricky Fold gives the same results as the Alternating Way Fold.

More recently, I have worked out some variations on the theme of removing no paper. These follow.

Square Slash-Slit

(Repeat "slash-slit" out loud quickly at your peril.) It is pleasing that a flexagon can be constructed from a square, or rectangle, with a single slit.

It must be a slash along the diagonal, however (Figure 4). The model will show four faces. There are two versions, the second allowing some presentational by-play. Use of paper money is recommended.

Using the Alternating Way Fold, flexing is not continuous, and two of the sides are just a little tricky to find. The designs are these: checkerboard; checkerboard; solid white; triangle in each corner, leaving a white square in the center.

Using the Same Way Fold, flexing is not possible without an additional move. The obvious way is to open up each side of the slit, and then move the two little flaps inside the slit to a position outside it. Then flexing is possible. Or, once you understand what is required, you can tuck the flaps properly as you make the valley folds. However, and best of all, the move can be done quite secretly in the course of doing the first flex. Being sure you have the right axis on which to fold the model in half; do the folding, but pull gently as you do it, and the little flaps will be adjusted automatically and properly, to be seen only at the very end of the flexing. Obviously, you can create some mischief by folding, flexing and unfolding, then challenging the knowledgeable paperfolder to duplicate the action. The flexing is continuous. The faces are these: solid white; two black (and six white) triangles opposite each other; two white triangles opposite each other, arranged differently from the previous two; a second solid white.

Square X-Slit

These models are made from squares with two intersecting slits making an X (Figure 5). Many versions of these are possible. Here are four of them. All have six faces.

Use of the Same Way Fold shows four faces continuously, and the other two easily, in the following manner: face 1, 2, 3, 4, 1, 5, 6, 4, 1, 2, etc. The faces are as follows: two black triangles opposite each other; two black triangles in a different arrangement; two white triangles arranged as in face 1; two white triangles, arranged as in face 2; solid white; solid black.

The Same Way Fold can also be used, with the inner flaps arranged differently: two opposite flaps creased one way, the other two creased the other way. The result flexes exactly the same as the one above, but with different designs: eight triangles of alternating color, circling about the center; two black triangles opposite each other; solid black; solid black; same as the second face; solid black. This does not appear to offer as much variation due to the repetition involved. However, the first face takes on a quite different appearance when flexed to the back of the model, as the eight

triangles are rearranged in an interesting way — four triangles of alternating color circle about the center, with one triangle in each corner of the model.

Use of the Alternating Way Fold creates a puzzle version. Four faces are shown continually, and the other two are deceptively hard to find, although no movement to a ring is required. The designs are these: checkerboard square; triangles in each corner so the center forms a white square; four large triangles alternating in color; checkerboard square; solid black; solid white. (This is my favorite of these four because the designs are handsome and the two solid color faces are a little tricky to find.)

The Alternating Way Fold can be used with the inner flaps arranged differently, also. The designs on one pair of faces are different, two triangles on each face, but differing in both color and arrangement.

Possibilities

Other flexagons can be constructed from the X-slit approach. For example, I constructed one to reveal the maximum of variety in design. This uses the fact that all flexagons show some faces more than once. The second and third showings of the face reorient parts of it. So although my multiversion shows only six faces, these reveal seven different designs: solid color; checkerboard of four squares; one triangle (out of eight); four faces each with different arrangements of three triangles.

Additionally, the methods given above can be combined to create still different design possibilities. For example, when a half of a corner is removed, instead of all or none of it, the design changes. For another example, combine a half-cross base with a slash-slit, and use the Same Way Fold. And so on.

This manuscript was submitted for possible publication after the Gathering for Gardner in January of 1993, but written more than a year or two before that. One of the items was taught at the Gathering and appeared in the Martin Gardner Collection.