How Random Are 3x + 1 Function Iterates?

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Abstract. The $3x+1$ problem concerns the behavior under iteration of the $3x+1$ function $T : \mathbb{Z} \to \mathbb{Z}$, defined by

$$T(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The $3x+1$ Conjecture asserts that any positive integer n eventually reaches 1 under iteration by T. Simple probabilistic models appear to model the average behavior of the initial iterates of a "random" positive integer n. This paper surveys results concerning more sophisticated probability models, which predict the behavior of "extreme" trajectories of $3x+1$ iterates. For example, the largest integer occurring in a trajectory starting from an integer n should be of size $n^{2(1+o(1))}$ as $n \to \infty$, and trajectories should be of length at most $41.677647\ldots \log\ n$ before reaching 1. The predictions of these models are consistent with empirical data for the $3x+1$ function.

1. Introduction

The well-known $3x+1$ problem concerns the behavior under iteration of the $3x+1$ function $T : \mathbb{Z} \to \mathbb{Z}$, defined by

$$T(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases} \tag{1.1}$$

The *3x+1 Conjecture* asserts that for each positive integer n there is some iterate k with $T^{(k)}(n) = 1$, where $T^{(2)}(n) = T(T(n))$, etc. The $3x+1$ Conjecture combines simplicity of statement with apparent intractability. Huge numbers of computer cycles have been expended studying it. In particular, it has now been verified for all $n < 2.702 \times 10^{16}$; see Oliveira e Silva [19].

The $3x+1$ problem appeared in Martin Gardner's "Mathematical Games" column in June 1972 [10]. Prior to that it circulated for many years by word of mouth. It is usually credited to Lothar Collatz, who studied problems similar to it in the 1930s, and who has stated that he circulated the problem in the early 1950s [6]. It was also independently discovered by B. Thwaites in 1952; see [22]. The first mathematical papers on it appeared around 1976 ([8], [21]), and now more than one hundred papers have been written about the problem. Surveys of the known results on the $3x + 1$ problem can be found in Lagarias [13], Müller [18], and Wirsching [25].

To describe iterates of the $3x+1$ function we introduce some terminology. The *trajectory* or *forward orbit* $\mathcal{O}(n)$ of the positive integer n is the sequence of its iterates

$$\mathcal{O}(n) := \left(n, T(n), T^{(2)}(n), T^{(3)}(n), ...\right). \tag{1.2}$$

Thus $\mathcal{O}(27) = (27, 41, 62, 31, 47, ...)$. The *total stopping time* $\sigma_\infty(n)$ of n is the minimal number of iterates k needed to reach 1, i.e.,

$$\sigma_\infty(n) := \min \left\{k : \ T^{(k)}(n) = 1\right\}, \tag{1.3}$$

where $\sigma_\infty(n) = \infty$ if no iterate equals 1. We let $t(n)$ denote the largest value reached in the trajectory of n, so that

$$t(n) := \sup \left\{T^{(k)}(n) \ : \ k \geq 0\right\}, \tag{1.4}$$

where $t(n) = \infty$ if the sequence of iterates is unbounded. Finally we let $r(n)$ count the fraction of odd iterates in a trajectory up to the point that 1 is reached, i.e.,

$$r(n) := \frac{1}{\sigma_\infty(n)} \# \left\{k \ : T^{(k)}(n) \equiv 1 \pmod{2} \text{ for } 0 \leq k \leq \sigma_\infty(n)\right\}. \tag{1.5}$$

The $3x + 1$ Conjecture is made plausible by the observation that $3x+1$ iterates behave "randomly" in some average sense. This randomness property can be formalized by taking an input probability distribution on the integers and looking at the probability distribution resulting from iterating the $3x + 1$ function some number of times. For example, if one takes the uniform distribution on $[1, 2^k]$, then the distribution of the k-vector

$$v_k(n) := \left(n, T(n), \ldots, T^{(k-1)}(n)\right) \pmod{2}$$

is exactly uniform, i.e., each possible binary pattern occurs exactly once in $1 \leq n \leq 2^k$. Furthermore the resulting pattern is periodic in n with period

2^k. This basic property was independently discovered by Everett [8] and Terras [21].

One may summarize this by saying that the parity of successive iterates of T initially behaves like independent coin flips. Since a number n is multiplied by $\frac{1}{2}$ if it is even and approximately $\frac{3}{2}$ if it is odd, one expects that on average it changes multiplicatively by their geometric mean, which is $(\frac{3}{4})^{1/2}$. Since this is less than 1, one expects the iterates to decrease in size and eventually become periodic.

Several heuristic probabilistic models describing $3x+1$ iterates are based on this idea; see [9], [13], [15], [20], [24]. The simplest of them assumes that this independent coin-flip behavior persists until 1 is reached under iteration. These mathematical models predict that the expected size of a "random" n after k steps should be about $(\frac{3}{4})^{k/2}n$, so that the average number of steps to reach 1 should be

$$\left(\frac{1}{2}\log\frac{4}{3}\right)^{-1}\log n \approx 6.95212\log n\ . \tag{1.6}$$

Furthermore, the number of steps $\sigma_\infty(n)$ for n to iterate to 1 should be normally distributed with mean $\left(\frac{1}{2}\log\frac{4}{3}\right)^{-1}\log n$ and variance $c_1\sqrt{\log n}$, for an explicit constant c_1; see [20], [24]. There is excellent numerical agreement of this model's prediction with $3x+1$ function data.

The pseudorandom character of $3x+1$ function iterates can be viewed as the source of the difficulty of obtaining a rigorous proof of the $3x+1$ Conjecture. On the positive side, Cloney, et al. [5] proposed using the $3x+1$ function as a pseudorandom number generator. A well-developed theory of pseudorandom number generators shows that even a single bit of pseudorandomness can be inflated into an arbitrarily efficient pseudorandom number generator; cf. Lagarias [14] and Luby [17]. Even though $3x+1$ function iterates possess quite a bit of structure (cf. Garner [11] and Korec [12]) they also seem to possess some residual structurelessness, which may be enough for a pseudorandom bit to be extracted.

From this viewpoint it becomes interesting to determine how well the properties of $3x+1$ iterates can be described by a stochastic model. In doing so we are in the atypical situation of modeling a purely deterministic process with a probabilistic model. Here we survey some recent results on stochastic models, obtained in joint work with Alan Weiss and David Applegate, which concern extreme behaviors of $3x+1$ iterates.

Lagarias and Weiss [15] studied two different stochastic models for the behavior of $3x+1$ function iterates. These models consist of a repeated random walk model for forward iterates by T, and a branching random walk model for backward iterates of T. The repeated random walk model results of [15] show almost perfect agreement between empirical data for $3x + 1$ function iterates for n up to 10^{11}. This model has the drawback that it does not model the fact that $3x + 1$ trajectories are not independent; indeed actual $3x+1$ trajectories coalesce to form a tree structure. The local structure of backward $3x+1$ iterates can be explicitly described using $3x+1$ trees, which are defined in Section 3. Lagarias and Weiss introduced a family of branching random walk models to describe the ensemble of such trees. These models make a prediction for extreme values of $\sigma_\infty(n)$ that is shown to coincide with that made by the repeated random walk model. That is, for these probabilistic models the deviation from independence exhibited by $3x + 1$ trajectories does not affect the asymptotic maximal length of extremal trajectories.

More recently Applegate and Lagarias ([1] and [3]) studied properties of the ensemble of all $3x + 1$ trees. They compared empirical data with predictions made from stochastic models, and found small systematic deviations of the distribution of $3x + 1$ inverse iterates from that predicted by the branching random walk model above. They observed that the distribution of the largest and smallest number of leaves possible in such trees of depth k appears to be narrower than what would be predicted by the model of Lagarias and Weiss. This leads to two conjectures stated in Section 4. It remains a challenge to exploit such regularities to obtain new rigorous results on the $3x + 1$ problem.

The $3x + 1$ Conjecture remains unsolved and is viewed as intractable. Various authors have obtained rigorous results in the direction of the $3x+1$ Conjecture, consisting of lower bounds for the number of integers n below a value x that have some iterate $T^{(k)}(n) = 1$. More generally, one may estimate for a positive integer a the quantity

$$\pi_a(x) := \text{card}\left\{n : 1 \le n \le x \text{ with some } T^{(k)}(n) = a\right\}.$$

It has been conjectured that for each positive $a \equiv 1$ or $2 \pmod{3}$ there is a positive constant c_a such that

$$\pi_a(x) > c_a x \quad \text{for all} \quad x \ge a\,;$$

see Applegate and Lagarias [2] and Wirsching [25]. At present the best rigorous bound of this sort states that for positive $a \equiv 1$ or $2 \pmod{3}$ one has

$$\pi_a(x) > x^{0.81}$$

for all sufficiently large x; see [2].

2. Repeated Random Walk Model

Lagarias and Weiss [15] studied a repeated random walk model for forward iterates by T. In this model, the successive iterates $\log T^{(k)}(n)$ are modeled by starting at $\log n$ and taking independent steps of size $-\log 2$ or $\log \frac{3}{2}$ with probability $\frac{1}{2}$ each. The random variable $\sigma_\infty^*(n)$ is the step number at which 0 is crossed. This random variable is finite with probability one, and the expected size of $\sigma_\infty^*(n)$ is given by (1.6). For each initial value n an entirely new random walk is used; this explains the name "repeated random walk model." What is the extremal size of $\sigma_\infty^*(n)$? Large deviation theory shows that with probability one

$$\limsup_{n\to\infty} \frac{\sigma_\infty^*(n)}{\log n} = \gamma_{RW} \simeq 41.677647 \ . \tag{2.1}$$

The constant γ_{RW} is the unique solution with $\gamma > (\frac{1}{2}\log\frac{4}{3})^{-1}$ of the functional equation

$$\gamma g\left(\frac{1}{\gamma}\right) = 1 \ , \tag{2.2}$$

where

$$g(a) := \sup_{\theta\in\mathbb{R}} \left[a\theta - \log \frac{1}{2}\left(2^\theta + \left(\frac{2}{3}\right)^\theta\right)\right] . \tag{2.3}$$

Next, let $t^*(n)$ equal the maximal value taken during the random walk, so that $t^*(n) \geq \log n$. Large deviation theory shows that, with probability 1,

$$\limsup_{n\to\infty} \frac{t^*(n)}{\log n} = 2 \ . \tag{2.4}$$

Finally, we consider the random variable

$$r^*(n) := \frac{1}{\sigma_\infty(n)} \# \left\{ k : T^{(k)}(n) > T^{(k-1)}(n) \text{ with } 1 \leq k \leq \sigma_\infty(n) \right\} .$$

Large deviation theory predicts that with probability 1 it satisfies

$$\limsup_{n\to\infty} r^*(n) = \rho \simeq 0.609090 \, .$$

The quantity ρ corresponds to an estimate of the maximum fraction of elements in a $3x+1$ trajectory that are odd. Large deviation theory also asserts that the graph of the logarithmically scaled trajectories

$$\left\{ \left(\frac{k}{\log n}, \frac{\log T^{(k)}(n)}{\log n} \right) : 0 \leq k \leq \sigma_\infty(n) \right\} \tag{2.5}$$

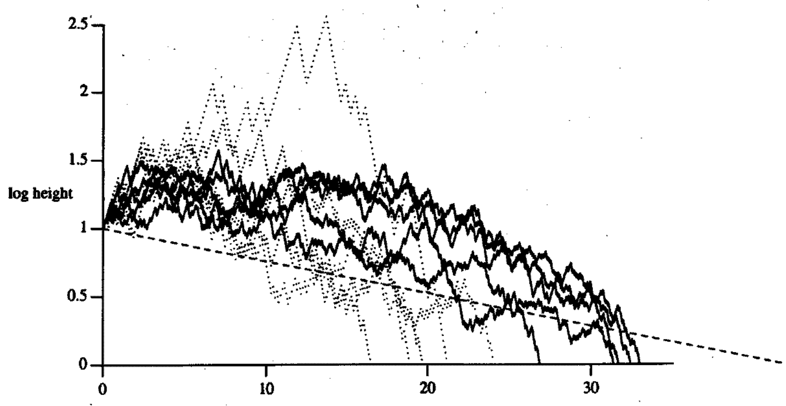

Figure 1. Scaled trajectories of n_k maximizing $\gamma(n)$ in $10^k \leq n \leq 10^{k+1}$ (dotted for $1 \leq k \leq 5$; solid for $6 \leq k \leq 10$).

approaches a characteristic shape. For the extremal trajectories for $\gamma^*(n)$ in (2.1) it is a straight line segment with slope -0.0231 starting from $(0,1)$ and ending at $(\gamma, 0)$; see Figure 1. For the extremal trajectories for $t^*(n)$ in (2.4), the limiting graph of these trajectories has a different appearance. The length of such trajectories approaches the limiting value $\sigma_\infty(n) \asymp 21.55 \log n$, and the graph (2.5) approximates the union of two line segments, the first with slope 0.1318 for $1 \leq k \leq 7.645 \log n$, starting from $(0,1)$, and ending at $(7.645, 2)$, and the second with slope -0.1439 for $7.645 \log n \leq k \leq 21.55 \log n$, starting at $(7.645, 2)$ and ending at $(21.55, 0)$; see Figure 2.

A comparison of (2.1) with numerical data up to 10^{11} is given in Table 1. It gives the longest trajectory, where $\gamma(n) = \frac{\sigma_\infty(n)}{\log n}$, and where $r(n)$ gives the fraction of odd entries in the iterates $(n, T(n), \ldots, T^{(k)}(n))$ in this trajectory.

The trajectories of these values of n are plotted in Figure 1, and the large deviations extremal trajectory is indicated by a dotted line. The agreement of the data up to 10^{11} with the repeated random walk model is quite good; an analysis in [15, Section 5] shows that in 10^{11} trials one should only expect to find a largest $\gamma(n) \approx 30$.

More recently V. Vyssotsky [23] has found much larger numbers with high values of $\gamma(n)$. He found that n =12,769,884,180,266,527 has $\sigma_\infty(n) = 1271$ and $\gamma(n) = 34.2716$, and that

$$n = 37{,}664{,}971{,}860{,}959{,}140{,}595{,}765{,}286{,}740{,}059$$

has $\sigma_\infty(n) = 2565$ and $\gamma(n) = 35.2789$. He also found a number around 10^{110} with $\gamma(n)$ exceeding 36.40.

k	n	$\sigma_\infty(n)$	$\gamma(n)$	$r(n)$
1	27	70	21.24	0.5857
2	703	108	16.48	0.5741
3	6,171	165	18.91	0.5818
4	52,527	214	19.68	0.5841
5	837,799	329	24.13	0.5927
6	8,400,511	429	26.91	0.5967
7	63,728,127	592	32.94	0.6030
8	127,456,254	593	31.77	0.6020
9	4,890,328,815	706	31.64	0.6020
10	13,371,194,527	755	32.38	0.6026
Random walks model			41.68	0.6091

Table 1. Maximal value of $\gamma(n)$ in intervals $10^k < n \leq 10^{k+1}$, $1 \leq k \leq 10$.

In Table 2 we present empirical results comparing the maximal value of $t(n)$ in intervals $10^k \leq n < 10^{k+1}$ up to 10^{11} with the prediction of (2.4). Here $\sigma_{\max}(n)$ denotes the number of steps taken until the maximum value is reached. Extensions of the data can be found in Leavens and Vermeulen [16]. The agreement with the stochastic model is quite good.

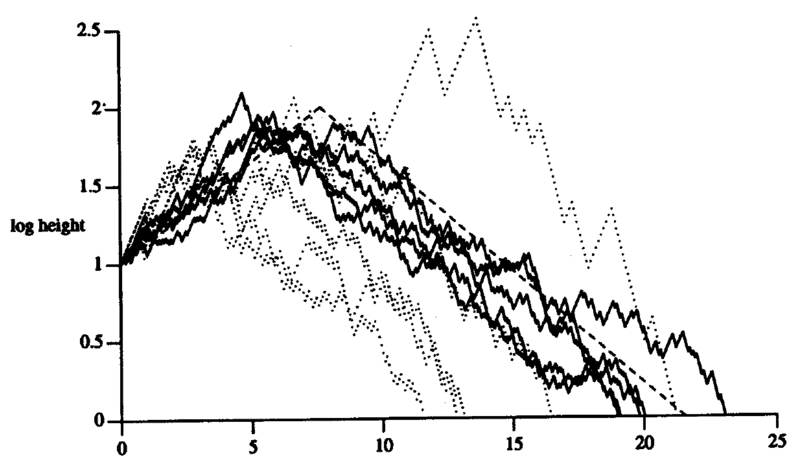

Figure 2. Scaled trajectories of n_k maximizing $(\log t(n))/\log n$ in $10^k \leq n \leq 10^{k+1}$ (dotted for $1 \leq k \leq 5$; solid for $6 \leq k \leq 10$).

k	n	$\frac{\log t(n)}{\log n}$	$\frac{\sigma_{\max}(n)}{\log n}$	$\gamma(n)$
1	27	2.560	13.65	21.24
2	703	1.791	7.32	16.48
3	9,663	1.790	2.83	12.86
4	77,671	1.819	3.46	13.14
5	704,511	1.788	2.75	11.59
6	6,631,675	1.976	5.86	23.05
7	80,049,391	1.903	5.06	19.84
8	319,804,831	2.099	4.65	19.10
9	8,528,817,511	1.909	5.20	20.03
10	77,566,362,559	1.897	6.86	19.02
Random walks model		2.000	7.645	21.55

Table 2. Largest value of $\frac{\log t(n)}{\log n}$ for $10^k \leq n \leq 10^{k+1}$, $1 \leq k \leq 10$.

In Figure 2 we plot the graphs of these extremal trajectories on a double logarithmic scale. The large deviations extremal trajectory is indicated by a dotted line. The deviations from the stochastic model are well within the range that would be expected from the size of the expected standard deviation for such models.

3. Branching Random Walk Model

Lagarias and Weiss [15] modeled backward iteration of the $3x+1$ function by a branching random walk.

In iterating the multivalued function T^{-1}, it proves convenient to restrict the domain of T to integers $n \not\equiv 0 (\bmod\ 3)$. In this case,

$$T^{-1}(n) := \begin{cases} \{2n\} & \text{if } n \equiv 1,\ 4,\ 5, \text{ or } 7 \pmod{9}, \\ \left\{2n, \frac{2n-1}{3}\right\} & \text{if } n \equiv 2 \text{ or } 8 \pmod{9}. \end{cases} \tag{3.1}$$

The restriction to nodes $n \not\equiv 0 (\bmod\ 3)$ is made because nodes $n \equiv 0 (\bmod\ 3)$ never branch, so do not have any significant effect on the size of the tree; see [15] for more explanation.

The set of inverse iterates associated to any n has a tree structure, where the branching at a node in the tree is determined by the node label (mod 9), using (3.1). In [15] these trees were called *pruned 3x+1 trees* because all nodes corresponding to $n \equiv 0 (\bmod\ 3)$ have been removed. Figure 3 shows pruned $3x+1$ trees of depth $k = 5$ starting from the root nodes $n = 7$ and $n = 20$; these trees have the minimal and maximal number of leaves possible for depth $k = 5$, respectively. In what follows we use the term *3x+1 tree* to mean pruned $3x+1$ tree.

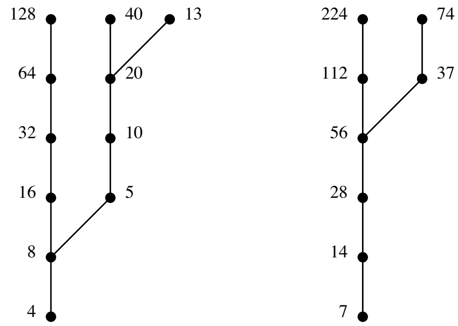

Figure 3. Pruned $3x + 1$ trees of depth $k = 5$ starting from root nodes $n = 4$ and $n = 7$.

The branching structure of a $3x + 1$ tree of depth k is completely determined by its root node $n \pmod{3^{k+1}}$, hence there are at most $2 \cdot 3^k$ possible distinct $3x+1$ trees of depth k. In fact, there are duplications, and the number $R(k)$ of distinct edge-labeled $3x + 1$ trees of depth k seems to grow empirically like θ^k, where $1.87 < \theta < 1.93$. (See Table 3 in Section 4.)

The problem of determining the largest trajectory to get to 1 under iteration in a $3x + 1$ tree is equivalent to studying the leaf in a $3x + 1$ tree having the smallest value, where the values of the inverse image nodes of a node of value n are $\{2n\}$ or $\{2n, \frac{2n}{3}\}$ depending on whether the tree has one or two branches, and the root node is assigned the value 1. We assign labels 0 or 1 to the edges, so that an edge from n to $2n$ is labeled 0 and an edge from n to $\frac{2n}{3}$ is labeled 1. The branching random walk arises from associating a walk to each path in the tree from a root node to a leaf node at depth k, where the random walk starts at the origin on the real axis at the root node and takes a step $\log 2$ along an edge labeled 0 and a step $\log \frac{2}{3}$ along an edge labeled 1. The random walk at a given leaf node ends at a position u on the real line, and the value assigned to that vertex by the rules above is exactly e^u. Lagarias and Weiss [15, Theorem 3.4] show that the expected size of $\log n^*(k)$, where $\log n^*(k)$ is the smallest value taken among all leaf nodes of depth k of a tree, satisfies (with probability one)

$$\lim_{k\to\infty} \frac{E[\log n^*(k)]}{\log k} = \gamma_{BP} \simeq 41.677647\ldots \ . \tag{3.2}$$

This constant $\gamma_{BP} = \beta^{-1}$, where β is the unique positive number satisfying $g^*(\beta) = 0$, and where

$$g^*(a) := \sup_{\theta\le 0} \left[a\theta - \log(2^\theta + \frac{1}{3}\left(\frac{2}{3}\right)^\theta \right]. \tag{3.3}$$

This constant is provably the same constant as (2.1), i.e., $\gamma_{BP} = \gamma_{RW}$; cf. [15, Theorem 4.1].

In order to prove the result (3.2), it was necessary to determine the distribution of the number of leaves in a tree in the branching process. In [15] it was shown that the expected number of leaves of a tree of depth k is $(\frac{4}{3})^k$, and as $k \to \infty$ the leaf probability distribution is of the form $W(\frac{4}{3})^k$, where W is a random variable satisfying

$$\text{prob}\{a < W < b\} = \int_a^b w_k(x)dx \ , \quad \text{for} \quad 0 < a < b < \infty \ , \tag{3.4}$$

and $w_k(x)$ is strictly positive on $(0, \infty)$; see [15, Theorem 3.2].

4. Distribution of 3x + 1 Trees

Applegate and Lagarias ([1] and [3]) studied properties of the ensemble of all $3x + 1$ trees. Here the *pruned 3x+1 tree* $\mathcal{T}_k(a)$ is the tree of inverse iterates of the $3x+1$ function grown backward to depth k, from a given root node a, with all nodes that have a label congruent to $0 \pmod 3$ removed.

Applegate and Lagarias [1] empirically studied the extremal distribution of the number of leaves in a $3x + 1$ tree of depth k as the root node is varied. Let $N^-(k)$ and $N^+(k)$ denote the minimal and maximal number of leaves, respectively. Since the number of leaves is expected to grow like $(\frac{4}{3})^k$, they studied the *normalized extreme values*

$$D^+(k) := \left(\frac{3}{4}\right)^k N^-(k) \quad \text{and} \quad D^-(k) := \left(\frac{3}{4}\right)^k N^+(k) \, . \tag{4.1}$$

Table 3 gives data up to $k = 30$. In this table, $R(k)$ counts the number of distinct edge-labeled $3x + 1$ trees of depth k. Based on this data, they proposed that the normalized extreme quantities remain bounded. This can be formalized as

Conjecture C$^{\#}$. *Let*

$$C^- := \liminf_{k \to \infty} D^-(k) \text{ and } C^+ := \limsup_{k \to \infty} D^+(k) \, . \tag{4.2}$$

Then these quantities satisfy the inequalities

$$0 < C^- < 1 < C^+ < \infty \, .$$

Applegate and Lagarias [3] compared this conjecture with predictions derived from one of the branching random walk models of [15]. The branching process is a multi-type Galton–Watson process with six types

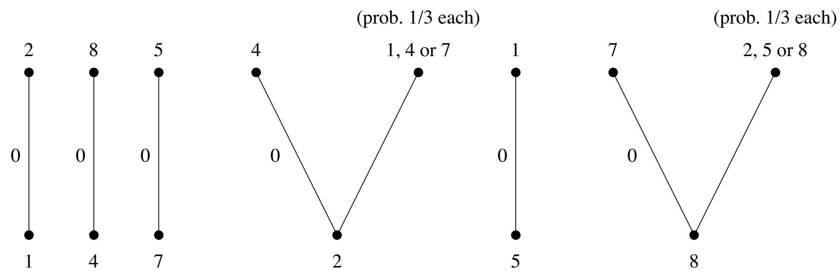

Figure 4. Transitions of the branching process $\mathfrak{B}$ [9]. The parent (bottom) always yields a child by the map $n \mapsto 2n$ (edge label 0), and it yields a second child by the multivalued map $n \mapsto \frac{1}{3}(2n-1)$ (edge label 1) if $n = 2$ or $8 \bmod 9$.

labeled 1, 2, 4, 5, 7, 8 pictured in Figure 4. (See Athreya and Ney [4] for a general discussion of such processes.)

The random walk arises from assigning the labels 0 and 1 to the edges of the resulting tree, where 0 represents the entering vertex being even and 1 represents the entering vertex being odd; the corresponding random walk takes a step $\log 2$ along an edge labeled 0 and takes a step $\log \frac{2}{3}$ along an edge labeled 1. The base vertex of the tree is located at the origin, so that each vertex of the tree corresponds to a random walk ending at some position u on the real line, and we consider the value of that vertex to be e^u. This value represents the initial value of the $3x+1$ iteration starting from that vertex of the tree. We make $\tilde{R}(k)$ independent draws of a tree of depth k generated by this process, choosing the root node uniformly $(\bmod\ 9)$, and choosing $\tilde{R}(k) = \Theta^k$ for a fixed constant $\Theta > 1$. (For actual $3x+1$ trees, $\tilde{R}(k) \le 3^{k+1}$ so $1 < \Theta \le 3$.) We consider as random variables the smallest and largest number of leaves that occur among this set of trees. Let $\tilde{N}^-(k)$ and $\tilde{N}^+(k)$ denote the expected value of the smallest number of leaves, respectively largest number of leaves. The analogues of normalized extreme values in the models are

$$\tilde{D}^-(k) := \left(\frac{3}{4}\right)^k \tilde{N}^-(k) \quad \text{and} \quad \tilde{D}^+(k) := \left(\frac{3}{4}\right)^k \tilde{N}^+(k) \ . \tag{4.3}$$

In [3] it is proved[1] that

$$\limsup \ \tilde{D}^+(k) = \infty$$

and

$$\liminf_{k\to\infty} \ \tilde{D}^-(k) = 0 \ .$$

[1]These facts are derived using the probability density for W given in (3.4).

					3x + 1 Function Trees		Branching Process	
k	$R(k)$	$N^-(k)$	$N^+(k)$	$\left(\frac{4}{3}\right)^k$	$D^-(k)$	$D^+(k)$	$\tilde{D}^-(k)$	$\tilde{D}^+(k)$
1	4	1	2	1.33	0.750	1.500	0.750	1.500
2	8	1	3	1.78	0.562	1.688	0.562	1.557
3	14	1	4	2.37	0.422	1.688	0.422	1.669
4	24	2	6	3.16	0.633	1.898	0.633	1.728
5	42	2	8	4.21	0.475	1.898	0.475	1.792
6	76	3	10	5.62	0.534	1.780	0.534	1.778
7	138	4	14	7.49	0.534	1.869	0.409	1.911
8	254	5	18	9.99	0.501	1.802	0.401	1.923
9	470	6	24	13.32	0.451	1.802	0.375	1.986
10	876	9	32	17.76	0.507	1.802	0.394	2.026
11	1638	11	42	23.68	0.465	1.774	0.352	2.054
12	3070	16	55	31.57	0.507	1.742	0.342	2.076
13	5766	20	74	42.09	0.475	1.758	0.307	2.118
14	10850	27	100	56.12	0.481	1.782	0.305	2.131
15	20436	36	134	74.83	0.481	1.791	0.300	2.166
16	38550	48	178	99.77	0.481	1.784	0.302	2.190
17	72806	64	237	133.03	0.481	1.782	0.289	2.211
18	137670	87	311	177.38	0.490	1.753	0.281	2.232
19	260612	114	413	236.50	0.482	1.746	0.273	2.255
20	493824	154	548	315.34	0.488	1.738	0.271	2.270
21	936690	206	736	420.45	0.490	1.751	0.268	2.292
22	1778360	274	988	560.60	0.489	1.762	0.265	2.310
23	3379372	363	1314	747.47	0.486	1.758	0.259	2.327
24	6427190	484	1744	996.62	0.486	1.750	0.255	2.344
25	12232928	649	2309	1328.83	0.488	1.738	0.252	2.360
26	23300652	868	3084	1771.77	0.490	1.741	0.249	2.375
27	44414366	1159	4130	2362.36	0.491	1.748	0.246	2.390
28	84713872	1549	5500	3149.81	0.492	1.746	0.243	2.405
29	161686324	2052	7336	4199.75	0.489	1.747	0.240	2.419
30	308780220	2747	9788	5599.67	0.491	1.748	0.238	2.433

Table 3. Normalized extreme values for $3x + 1$ trees and for the branching process.

These results for the branching random walk model do not agree with Conjecture C$^{\#}$ above. The number of leaves in actual $3x + 1$ trees empirically exhibits less variability than is predicted by this branching random walk model. The paper [3] presents more evidence in favor of Conjecture $C^{\#}$ and formulates additional conjectures concerning the average number of leaves in $3x+1$ trees that have a fixed node $n(\bmod 3^{k+1})$.

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