# Misfiring Tasks

Ken Knowlton

Years ago I went on a collecting trip, visiting non-innumeretic friends asking for contributions of a very particular kind. Martin Gardner gave me some help, not as much as I expected; others scraped together a token from here and there. I browsed, I pilfered, but the bottom of my basket ended up barely covered and, except for the accumulated webs and dust, so it remains. At this point, before throwing it out, I pass it around once more for contributions.

When a cold engine starts, it often misfires a few times before running smoothly. There are mathematical tasks like that: Parameterized in $n$, each can be accomplished for some hit-and-miss pattern of integer values until, after some last "misfiring" value, things run smoothly, meaning simply that the task can be performed for all higher integers. One such task is to partition a square into $n$ subsquares (i.e., using all of its area, divide it into $n$ non-overlapping subsquares). This can in fact be performed for $n = 4, 6, 7, 8, \ldots$ where 5 is "missing" from the sequence.

A devilishly difficult kind of mathematical problem, I thought, might be to pose such a question in reverse, e.g., What is a task parameterized in $n$ that last misfires for $n = 6$?, or for $n = 9$?, etc. More precisely, a task that last misfires for $N$

- can be done for at least one earlier $n$ $(1 < n < N)$;
- cannot be done for $n = N$;
- can be done for all $n > N$;

and, to be sporting,

- the task must not be designed in an ad hoc way with the desired answer built in — must not, for example, involve an equation with poles or zeros in just the right places;
- it must be positively stated, *not* for example "prove the impossibility of . . .";
- it must at least seem to have originated innocently, not from a generative construct such as "divide $n$ things into groups of 17 and 5."

For a mathematician or logician, these are attrociously imprecise specifications. But here, for me at least, lies the intriguing question: Will we agree that one or another task statement lies within the spirit of the game? This is, of course, a sociological rather than mathematical question; my guess is that the matter lies somewhere on the non-crispness scale between agreement on "proof" and agreement on "elegance."

Meager as the current collection is, its members do exhibit a special charm. I have answers only for $N = 5$, 6, 7, 9, 47, and 77. Some of these are well known, others quite esoteric:

- (5) Stated above, well known, with a rather obvious proof: Partition a square into $n$ subsquares.
- (6) Divide a rectangle into $n$ disjoint subrectangles without creating a composite rectangle except for the whole (Frank Sinden).
- (7) Design a polyhedron with $n$ edges. (Generally known. The tetrahedron has 6 edges, the next have 8, 9, … edges.)
- (9) Cut a square into $n$ acute triangles with clean topology: no triangle's vertex may lie along the side of another triangle. Possible for 8, 10, 11, 12 …. (Charles Cassidy and Graham Lord, even after developing a complete proof, ask themselves "Why is 9 missing?")
- (9) Place $n$ counters on an infinite chessboard such that each pair exhibits different numbers of row-mates and/or different numbers of column-mates. (Impossible for $n = 2$, 5, or 9; conjectured by Ken Knowlton, proved by Ron Graham. One of Martin Gardner's books contains essentially this problem as a wire-identification task, but the stated answer implicitly but erroneously suggests that the task can be performed for any $n$).
- (47) Cut a cube into $n$ subcubes (called the "Hadwiger" problem, tracked for years by Martin Gardner, finally clinched independently by Doris Rychener and A. Zbinder who demonstrated a dissection into 54 subcubes).
- (77) Partition $n$ into distinct positive integers whose reciprocals sum to one, e.g., such a partition for 11 is 2, 3, 6 since $2 + 3 + 6 = 11$ and $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$. (Ron Graham is the only person I know who would think of such a problem, and who did, and who then went on to prove that 77 is the largest integer that cannot be so partitioned.)

What we have here is an infinite number of problems of the form "Here's the answer, what's the question?" Usually such a setup is too wide open to be interesting. But a misfire problem, as I think I have defined it, has such a severely constrained answer that it is almost impossibly difficult. But "difficult" may be the wrong word. The trouble is that a misfire problem is a math problem with no implied search process whatever, except to review all the math and geometry that you already know. Instead of searching directly for a task last misfiring, say, for $n = 13$, it's much more likely that in the course of your mathematical ramblings you will someday bump into one.

I invite further contributions to the collection. Or arguments as to why this is too mushily stated a challenge.