Puzzles from Around the World

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Introduction

Most of the puzzles in this collection were presented in the Logigram, a company newsletter published at Logicon, Inc., where I worked for 27 years. The regular problem column appeared from 1984 to 1994 and was called "Puzzles from Around the World"; it consisted of problems from a number of sources: other problem columns, other solvers through word of mouth, embellishments or adaptations of such problems, or entirely new problems of my own creation. These problems are for the enjoyment of the solver and should be passed on to others for their enjoyment as well.

The problems are arranged in three general categories of *easy*, *medium*, and *hard*; the solution section gives an approach to answering each of the problems. A sources section provides information on the source of each problem as far as I know. I would be delighted to hear from anyone who can improve the solution approach, add information on the source of the problem, or offer more interesting problems for future enjoyment.

I thank the following people for providing problems and ideas that have contributed to the richness of the problems involved: Leon Bankoff, Brian Barwell, Nick Baxter, Laurie Brokenshire, James Dalgety, Clayton Dodge, Martin Gardner, Dieter Gebhardt, Allan Gottlieb, Yoshiyuki Kotani, Harry Nelson, Karl Scherer, David Singmaster, Naoaki Takashima, Dario Uri, Bob Wainwright, and, especially, Nob Yoshigahara. Source information begins on page 82.

Easy Problems

E1. A man has breakfast at his camp. He gets up and travels due North. After going 10 miles in a straight line he stops for lunch. After lunch he gets up and travels due North. After going 10 miles in a straight line he finds himself back at camp. Where on earth could he be?

E2. Find the rule for combining numbers as shown below (e. g., 25 and 9 combine to give 16) and use the rule to determine x.

	63		9		38		33		32		12	
88		25		16		18		15		x		5

E3. What number belongs at x in this sequence?

	34		32		36		46		64		75		50		35		34	
16		18		14		22		x		40		35		15		20		12

E4. It is approximately 2244.5 nautical miles from Los Angeles to Honolulu. A boat starts from being at rest in Los Angeles Harbor and proceeds at 1 knot per hour to Honolulu. How long does it take?

E5. You have containers that hold 15 pints, 10 pints, and 6 pints. The 15-pint container starts out full, and the other two start out empty: (15, 0, 0). Through transferring liquid among the containers, measure exactly two pints for yourself to drink and end up with 8 pints in the 10-pint container and 5 pints in the 6-pint container. Find the most efficient solution.

E6. You are at a lake and have two empty containers capable of holding exactly $\pi\,(= 3.14159...)$ and $e\,(= 2.7182818...)$ liters of liquid. How many transfers of liquid will it take you to get a volume of liquid in one container that is within one percent of exactly one liter?

E7. Calculate the expansion of this 26 term expression:

$$E = (x-a)(x-b)(x-c)(x-d)...(x-y)(x-z)$$

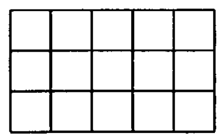
E8. Find a nine-digit number made up of 1, 2, 3, 4, 5, 6, 7, 8, 9 in some permutation such that when digits are removed one at a time from the right the number remaining is divisible in turn by 8, 7, 6, 5, 4, 3, 2 and 1.

E9. My regular racquetball opponent has a license plate whose three-digit part has the following property. Divide the number by 3, reverse the digits of the result, subtract 1 and you produce the original number. What is the number and what is the next greater number (possibly with more than three digits) having this property?

E10. Find nine single-digit numbers other than (1, 2, 3, ..., and 9) with a sum of 45 and a product of $9! = 362{,}880$.

E11. A knight is placed on an infinite checkerboard. If it cannot move to a square previously visited, how can you make it unable to move in as few moves as possible?

E12. Place the numbers from 1 to 15 in the 3×5 array so that each column has the same sum and each row has the same sum.



E13. The Bridge. Four men must cross a bridge. They all start on the same side and have 17 minutes to get across. It is night, and they need their one flashlight to guide them on any crossing. A maximum of two people can cross at one time. Each man walks at a different speed: A takes 1 minute to cross; B takes 2 minutes; C takes 5 minutes, and D takes 10 minutes. A pair must walk together at the rate of the slower man's pace. Can all four men cross the bridge? If so, how?

Try these other problems.

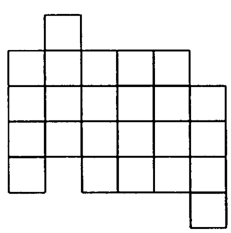
(a) There are six men with crossing times of 1, 3, 4, 6, 8, and 9 minutes and they must cross in 31 minutes.

(b) There are seven men with crossing times of 1, 2, 6, 7, 8, 9, and 10 minutes, and the bridge will hold up to three men at a time, and they must cross in 25 minutes.

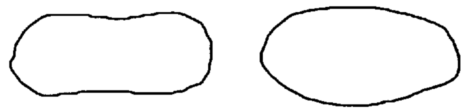
E14. Divide the figure below into

(a) 4 congruent pieces and

(b) 3 congruent pieces.



E15. Potato Curves. You are allowed to draw a closed path on the surface of each of the potatoes shown below. Can you draw the two paths so that they are identical to each other in three-dimensional space?



E16. Suppose a clock's second hand is exactly on a second mark and exactly 18 second marks ahead of the hour hand. What time is it?

E17. Shoelace Clock. You are given some matches, a shoelace, and a pair of scissors. The lace burns irregularly like a fuse and takes 60 minutes to burn from end to end. It has a symmetry property in that the burn rate a distance x from the left end is the same as the burn rate the same distance x from the right end. What is the minimum time interval you can measure?

Medium Problems

M1. You have a 37-pint container full of a refreshing drink. N thirsty customers arrive, one having an 11-pint container and another having a $2N$-pint container. How will you most efficiently measure out 1 pint of drink for each customer to drink in turn and end up with N pints in the 11-pint container and $37\text{-}2N$ pints in the 37-pint container if

(a) $N = 3$;
(b) $N = 5$?

M2. Three points have been chosen randomly from the vertices of a cube. What is the probability that they form (a) an acute triangle; (b) a right triangle?

M3. You were playing bridge as South and held 432 in spades, hearts, and diamonds. In clubs you held 5432. Despite your lack of power you took 6 tricks, making a 4-club contract. Produce the other hands and a line of play that allows this to occur.

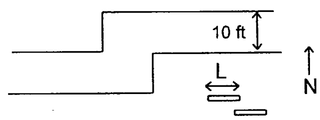
M4. From England comes the series … 35, 45, 60, x, 120, 180, 280, 450, 744, 1260, … Find a simple continuous function to generate the series and compute the surprise answer for x.

M5. How many ways can four points be arranged in the plane so that the six distances between pairs of points take on only two different values?

M6. From the USSR we are asked to simplify $x = \sqrt[3]{2 + \frac{10}{3\sqrt{3}}} + \sqrt[3]{2 - \frac{10}{3\sqrt{3}}}$.

M7. Find all primes p such that $2^p + p^2$ is also prime. Prove there are no others.

M8. The river shown below is 10 feet wide and has a jog in it. You wish to cross from the south to the north side and have only two thin planks of length L and width 1 ft to help you get across. What is the least value for L that allows a successful plan for crossing the river?



M9. A regular pentagon is drawn on ordinary graph paper. Show that no more than two of its vertices lie on grid points.

M10. 26 packages labeled A to Z are known to each weigh whole numbers of pounds in the range 1 to 26.

(a) Determine the weight of each package with a two-pan balance and four weights of your own design.
(b) Now do it with three weights.

M11. Music on the planet Alpha Lyra IV consists of only the notes A and B. Also, it never includes three repetitions of any sequence nor does the repetition BB ever occur. What is the longest Lyran musical composition?

M12. Many crypto doorknob locks use doors with five buttons numbered from 1 to 5. Legal combinations allow the buttons to be pushed in specific order either singly or in pairs without pushing any button more than once. Thus [(12), (34)] = [(21), (34)]; [(1), (3)]; and [(2), (13), (4)] are legal combinations while [(1), (14)]; [(134)]; and [(13), (14)] are not.

(a) How many legal combinations are there?
(b) If a sixth button were added, how many legal combinations would there be?

M13. Find the smallest prime number that contains each digit from 1 to 9 at least once.

M14. Dissect a square into similar rectangles with sides in the ratio of 2 to 1 such that no two rectangles are the same size. A solution with nine rectangles is known.

M15. Divide an equilateral triangle into three contiguous regions of identical shape if

(a) All three regions are the same size;
(b) all three regions are of different size;
(c) two of the regions are the same size and the third region is a different size.

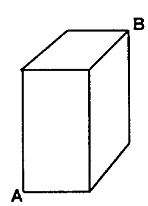
M16. Dissect a square into similar right triangles with legs in the ratio of 2 to 1 such that no two triangles are the same size. A solution with eight triangles is known.

M17. You and two other people have numbers written on your foreheads. You are told that the three numbers are primes and that they form the sides of a triangle with prime perimeter. You see 5 and 7 on the other two people, both of whom state that they cannot deduce the number on their own foreheads. What number is written on your forehead?

M18. A snail starts crawling from one end along a uniformly stretched elastic band. It crawls at a rate of 1 foot per minute. The band is initially 100 feet long and is instantaneously and uniformly stretched an additional 100 feet at the end of each minute. The snail maintains his grip on the band during the instant of each stretch. At what points in time is the snail (a) closest to the far end of the band, and (b) farthest from the far end of the band?

M19. An ant crawls along the surface of a $1 \times 1 \times 2$ "dicube" shown below.

(a) If the ant starts at point A, which point is the greatest distance away? (It is not B.)
(b) What are the two points farthest apart from each other on the surface of the dicube? (The distance, d, between these points is greater than 3.01).



M20. Humpty Dumpty. "You don't like arithmetic, child? I don't very much," said Humpty Dumpty.

"But I thought you were good at sums," said Alice.

"So I am," said Humpty Dumpty. "Good at sums, oh certainly. But what has that got to do with liking them? When I qualified as a Good Egg — many, many years ago, that was — I got a better mark in arithmetic than any of the others who qualified. Not that that's saying a lot. None of us did as well in arithmetic as in any other subject."

"How many subjects were there?" said Alice, interested.

"Ah!" said Humpty Dumpty, "I must think. The number of subjects was one third of the number of marks obtainable in any one subject. And I ought to mention that in no two subjects did I get the same mark, and that is also true of the other Good Eggs who qualified."

"But you haven't told me . . . ," began Alice.

"I know I haven't told you how many marks in all one had to obtain to qualify. Well, I'll tell you now. It was a number equal to four times the maximum obtainable in one subject. And we all just managed to qualify."

"But how many . . . ," said Alice.

"I'm coming to that," said Humpty Dumpty. "How many of us were there? Well, when I tell you that no two of us obtained the same assortment of marks — a thing which was only just possible — you'll be well on the way to the answer. But to make it as easy as I can for you, I'll put it another way. The number of other Good Eggs who qualified when I did, multiplied by the number of subjects (I've told you about that already), gives a product equal to half the number of marks obtained by each Good Egg. And now you can find out all you want to know."

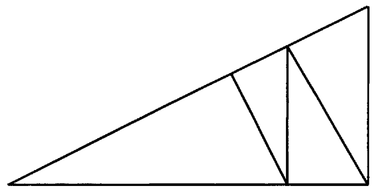
He composed himself for a nap. Alice was almost in tears. "I can't," she said, "do any of it. Isn't it differential equations, or something I've never learned?"

Humpty Dumpty opened one eye. "Don't be a fool, child," he said crossly. "Anyone ought to be able to do it who is able to count on five fingers."

What was Humpty Dumpty's mark in Arithmetic?

Hard Problems

H1. Similar Triangles. The figure following this problem shows a 30–60–90 triangle divided into four triangles of the same shape. How many ways can you find to do this? (Nineteen solutions are known.)

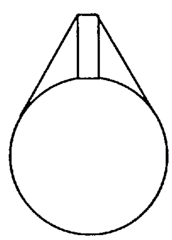


H2. "To reward you for killing the dragon," the Queen said to Sir George, "I grant you the land you can walk around in a day." She pointed to a pile of wooden stakes. "Take some of these stakes with you," she continued. "Pound them into the ground along your way, and be back at your starting point in 24 hours. All the land in the convex hull of your stakes will be yours." (The Queen had read a little mathematics.)

Assume that it takes Sir George 1 minute to pound a stake and that he walks at a constant speed between stakes. How many stakes should he take with him to get as much land as possible?

H3. An irrational punch centered on point *P* in the plane removes all points from the plane that are an irrational distance from *P*. What is the least number of irrational punches needed to eliminate all points of the plane?

H4. Imagine a rubber band stretched around the world and over a building as shown below. Given that the width of the building is 125 ft and the rubber band must stretch an extra 10 cm to accommodate the building, how tall is the building? (Use 20,902,851 ft for the radius of the earth.)



H5. A billiard ball with a small black dot, *P*, on the exact top is resting on the horizontal plane. It rolls without slipping or twisting so that its contact point with the plane follows a horizontal circle of radius equal to that of the ball. Where is the black dot when the ball returns to its initial resting place?

H6. Locate m points in the plane (perhaps with some exactly on top of others) so that each of them is a unit distance from exactly n of the others for (m,n) =

(a) (3, 2),
(b) (7, 4),
(c) (11, 6),
(d) (9, 4),
(e) (12, 4),
(f) (8, 3),
(g) (12, 5).

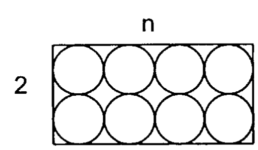
H7. A good approximation of π using just two digits is $\pi = 3.1$.

(a) Find the best approximation using two digits of your choice. You may use $+, -, \times, /$, exponents, decimal points, and parentheses. No square roots or other functions are allowed.
(b) The same as part (a) except square roots are allowed.
(c) Show that $1/\pi$ can be approximated with arbitrary accuracy using any two digits if the rules of part (b) are followed.

H8. (a) What region inside a unit square has the greatest ratio of area to perimeter?

(b) What volume inside a unit cube has the greatest ratio of volume to surface area?

H9. As shown below it is easy to place $2n$ unit diameter circles in a $2 \times n$ rectangle. What is the smallest value of n for which you can fit $2n + 1$ such circles into a $2 \times n$ rectangle?



H10. You are given two pyramids *SABCD* and *TABCD*. The altitudes of their eight triangular faces, taken from vertices S and T, are all equal to 1. Prove or disprove that line ST is perpendicular to plane *ABCD*.

H11. Tennis Paradox. Two evenly matched tennis players are playing a tiebreak set. The server in any game wins each point with a fixed probability p, where $0 < p < 1$. For what values of p and score situations during a set can the player ahead according to the score have less than a 50% chance of winning the set?

H12. My uncle's ritual for dressing each morning except Sunday includes a trip to the sock drawer, where he (1) picks out three socks at random, then (2) wears any matching pair and returns the odd sock to the drawer or (3) returns the three socks to the drawer if he has no matching pair and repeats steps (1) and (3) until he completes step (2). The drawer starts with 16 socks each Monday morning (eight blue, six black, two brown) and ends up with four socks each Saturday evening.

(a) On which day of the week does he average the longest time to dress?
(b) On which day of the week is he least likely to get a pair from the first three socks chosen?

H13. The hostess at her 20th wedding anniversary party tells you that the youngest of her three children likes her to pose this problem, and proceeds to explain: "I normally ask guests to determine the ages of my three children, given the sum and product of their ages. Since Smith missed the problem tonight and Jones missed it at the party two years ago, I'll let you off the hook." Your response is "No need to tell me more, their ages are . . . "

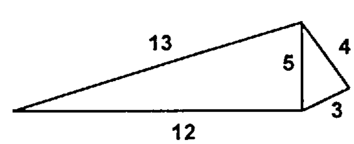
H14. Minimum Cutting Length. What is the minimum cut-length needed to divide

(a) a unit-sided equilateral triangle into four parts of equal area?
(b) a unit square, into five parts of equal area?
(c) an equilateral triangle into five parts of equal area?

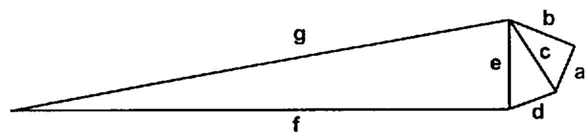
H15. A number may be approximated by a/b where a and b are integers. Define the goodness of fit of a/b to x as $g(a, b, x) = a|bx - a|$. An approximation to π is 355/113, with $g(355, 113, \pi) = 0.0107$.

(a) Find the minimum a and b so that $g(a, b, e) < 0.0107\ldots$, where $e = 2.71828\ldots$.
b) Find the minimum c and d so that $g(c, d, \pi) < 0.0107\ldots$.

H16. (a) The figure shows two Pythagorean triangles with a common side where three of the five side-lengths are prime numbers. Find other such examples.



(b) Can a third Pythagorean triangle be abutted such that 4 of the 7 lengths are primes?



H17. The inhabitants of Lyra III recognize special years when their age is of the form $a = p^2q$, where p and q are different prime numbers. The first few such special years are 12, 18 and 20. On Lyra III one is a student until reaching a special year immediately following a special year; one then becomes a master until reaching a year that is the third in a row of consecutive special years; finally one becomes a sage until death, which occurs in a special year that is the fourth in a row of consecutive special years.

(a) When does one become a master?
(b) When does one become a sage?
(c) How long do the Lyrans live?
(d) Do five or six special years ever occur consecutively?

H18. The planet Alpha Lyra IV is an oblate spheroid. Its axis of rotation coincides with the spheroid's small axis, just as one observes for Sol III. It's internal structure, however, is unique, being made of a number of coaxial right circular cylindrical layers, each of homogeneous composition. The common axis of these layers is the planet's rotational axis. The outermost of these layers is nearly pure krypton, and the next inner layer is an anhydrous fromage. Cosmic, Inc., is contemplating mining the outer layer, and the company's financial planners have found that the venture will be sound if there are more than one million cubic spandrals of krypton in the outer layer. (A spandral is the Lyran unit of length.)

Unfortunately, the little that is known about the krypton layer was received via sub-etherial communication from a Venerian pioneer immediately prior to its demise at the hands (some would say wattles) of a frumious snatcherband. The pioneer reported with typical Venerian obscurantism that the ratio of the volume of the smallest sphere that could contain the planet to the volume of the largest sphere that could be contained within the planet is 1.331 to 1. It (the Venerians are sexless) also reported that the straight line distance between it and its copod was 120 spandrals. (A copod corresponds roughly to something between a sibling and a rootshoot.) The reporting Venerian was mildly comforted because the distance to its copod was the minimum possible distance between the two. By nature the Venerians can only survive at the krypton/fromage boundary and by tragic mistake the two copods had landed on disconnected branches of the curve of intersection of the krypton/fromage boundary and the planetary surface.

Should Cosmic, Inc., undertake the mining venture?

H19. Taurus, a moon of a-Lyra IV (hieronymous), was occupied by a race of knife-makers eons ago. Before they were wiped out by a permeous accretion of Pfister-gas, they dug a series of channels in the satellite surface. A curious feature of these channels is that each is a complete and perfect circle, lying along the intersection of a plane with the satellite's surface. An even more curious fact is that Taurus is a torus (doughnut shape).

Five students of Taurus and its ancient culture were discussing their field work one day when the following facts were brought to light:

- The first student had dug the entire length of one of the channels in search of ancient daggers. He found nothing but the fact that the length of the channel was 30π spandrals.
- The second student was very tired from his work. He had dug the entire length of a longer channel but never crossed the path of the first dagger digger.
- The third student had explored a channel 50π spandrals in length, crossing the channel of the haggard dagger digger.
- The fourth student, a rather lazy fellow (a laggard dagger digger?), had merely walked the 60π spandral length of another channel, swearing at the difficulties he had in crossing the channel of the haggard dagger digger.
- The fifth student, a rather boastful sort, was also tired because he had thoroughly dug the entire length of the largest possible channel.

How long was the channel which the braggart haggard dagger digger dug?

H20. Define

$$F(n) = 2n + \frac{2}{3} - e^n \sum_{k=0}^{n-1} (k-n)^k e^{-k}/k!,$$

where $n = 1, 2, 3, \ldots$ and $e = 2.7182818\ldots$

(a) Prove that $F(n)$ goes to zero as n goes to infinity.
(b) Find $F(1000)$ to three significant figures.
(c) Find the smallest m such that the magnitude of $F(m)$ is less than the magnitude of $F(m+1)$.

Solutions to Easy Problems

E1. Anywhere within 10 miles of the north pole.

E2. At first glance, it appears that the rule might be subtraction, and $x = 17$. But this is not right because 18 is not the difference of 38 and 16. Instead, the rule is that two numbers combine to give a number that is the sum of the digits of the two numbers. Thus $x = 11$.

E3. In each case the numbers combine by summing the products of their digits. Thus $x = 4 \times 6 + 2 \times 2 = 28$.

E4. The trick here is that 1 knot = 1 nautical mile per hour, so 1 knot per hour is a constant acceleration of 1 nmi per hour per hour.
$2244.5 = \frac{at^2}{2}$ gives $t = 67$ hours.

E5. 15 moves: (15, 0, 0), (9, 0, 6), (9, 6, 0), (3, 6, 6), (3, 10, 2), (3, 10, 0), (3, 4, 6), (7, 0, 6), (7, 6, 0), (1, 6, 6), (1, 10, 2), (11, 0, 2), (11, 2, 0), (5, 2, 6), (5, 8, 0), (0, 8, 5).

E6. The goal is to find the smallest integer values of a and b so that

$$n = 2(a + b - 1)$$

is a minimum and $0.99 < ae - b\pi < 1.01$ or $0.99 < a\pi - be < 1.01$.

By numerical search we find

$$73\pi - 84e = 1.00059 \text{ and } 57e - 49\pi = 1.004024$$

are the smallest values satisfying the above equations. Thus $a = 57$, $b = 49$, and $n = 210$ transfers is the minimum.

E7. One of the terms is $(x - x)$, so $E = 0$.

E8. The number is 381,654,729.

E9. The license number is 741; the next greater number is 7,425,741.

E10. (1, 2, 4, 4, 4, 5, 7, 9, 9).

E11. The figure below shows how to trap the knight after 15 moves.

				1				
	3							
			2		0			
		4				14		
5				15				13
		6				12		
			8		10			
	7						11	
				9				

E12. See the array below.

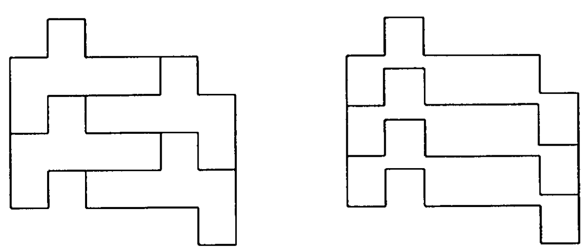
1	14	2	12	11
8	6	9	7	10
15	4	13	5	3

E13. Yes; *A* and *B* go across, *A* comes back; *C* and *D* go across, *B* comes back; *A* and *B* go across.

(a) 1 and 3 go across, 1 comes back; 8 and 9 go across, 3 comes back; 1 and 6 go across, 1 comes back; 1 and 4 go across, 1 comes back; 1 and 3 go across.

(b) 1 and 2 go across, 1 comes back; 8, 9, and 10 go across, 2 comes back; 1, 6,and 7 go across, 1 comes back; 1 and 2 go across.

E14. The solutions are shown in the following figures.



E15. Imagine intersecting the potatoes with each other. The path of their intersecting surfaces is a desired path.

E16. If the hour hand is exactly on a second mark then the second hand will always be on the 12. For the second hand to be 18 second marks ahead of the hour hand the hour hand must be at the 42^{nd} second mark, and the time is 8:24.

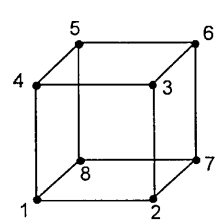
E17. Cut the lace in half, producing pieces A and B. Burn A from both ends, noting the point that burns last. Cut B at that corresponding point, producing pieces C and D. At the same time, start burning C from both ends and D from one end. When C is consumed, light the other end of D. In 3.75 minutes, D will finish burning. This is the shortest time interval that can be measured.

Solutions to Medium Problems

M1. (a) 32 moves; (37, 0, 0), (31, 0, 6), (31, 6, 0), (25, 6, 6), (25, 11, 1), (36, 0, 1), (36, 0, 0), (30, 0, 6), (30, 6, 0), (24, 6, 6), (24, 11, 1), (35, 0, 1), (35, 0, 0), (29, 0, 6), (29, 6, 0), (23, 6, 6), (23, 11, 1), (23, 11, 0), (23, 5, 6), (29, 5, 0), (29, 0, 5), (18, 11, 5), (18, 10, 6), (24, 10, 0), (24, 4, 6), (30, 4, 0), (30, 0, 4), (19, 11, 4), (19, 9, 6), (25, 9, 0), (25, 3, 6), (31, 3, 0)

(b) 40 moves; (37, 0, 0), (26, 11, 0), (26, 1, 10), (26, 0, 10), (36, 0, 0), (25, 11, 0), (25, 1, 10), (25, 0, 10), (35, 0, 0), (24, 11, 0), (24, 1, 10), (34, 1, 0), (34, 0, 0), (23, 11, 0), (23, 1, 10), (33, 1, 0), (33, 0, 0), (22, 11, 0), (22, 1, 10), (22, 0, 10), (11, 11, 10), (21, 11, 0), (21, 1, 10), (31, 1, 0), (31, 0, 1), (20, 11, 1), (20, 2, 10), (30, 2, 0), (30, 0, 2), (19, 11, 2), (19, 3, 10), (29, 3, 0), (29, 0, 3), (18, 11, 3), (18, 4, 10), (28, 4, 0), (28, 0, 4), (17, 11, 4), (17, 5, 10), (27, 5, 0).

M2. There are $7 \times 6/2 = 21$ choices using vertex 1. See the figure. The probability of an acute triangle is 3/21 or 1/7; the probability of a right triangle is 18/21 or 6/7, as seen from the list below. $A = 135, 137, 157$. $R =$ 123, 124, 125, 126, 127, 128, 134, 136, 138, 145, 146, 147, 148, 156, 158, 167, 168, 178.



M3. The hand below does the job if the play goes as follows. The first five tricks are alternate spade and heart ruffs by North and West, with East underruffing North each time. North wins the sixth trick with the 10 of diamonds, East playing the 9. North next leads a heart which West trumps with the queen. The next four tricks are won by South with the 5432 of clubs; North and East discard all their diamonds on these tricks. The final two tricks are won by South's long diamonds.

```
                         S -
                         H 98765
                         D 108765
                         C J97
S AKQJ1098765                                S -
H -                                          H AKQJ10
D -                      S 432               D AKQJ9
C AKQ                    H 432               C 1086
                         D 432
                         C 5432
```

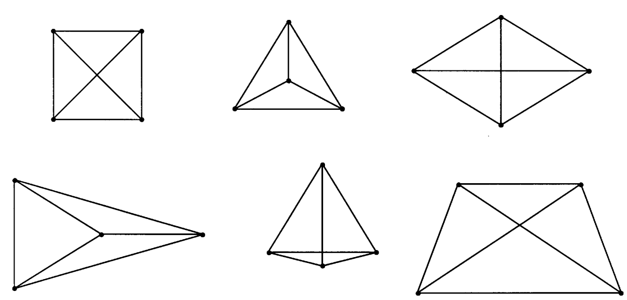
M4. The series can be expressed by a simple continuous function:

$$F(n) = 120(2^n - 1)/n, \text{for } n \neq 0.$$

To get x, we take the limit of $F(n)$ as n goes to 0.

$$x = 120 \ln 2 = 83.17766\ldots.$$

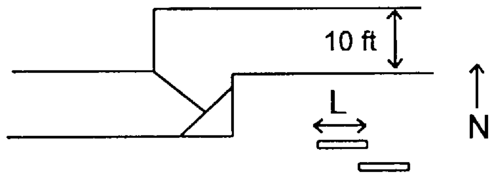
M5. There are six ways as shown below.



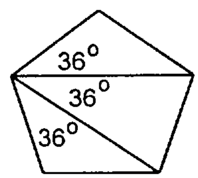
M6. $x^3 - 2x - 4 = 0$. The only real root is $x = 2$.

M7. For $p = 3$ we get the prime 17. For $p > 3$ we have $p = 6k + 1$ or $p = 6k - 1$. Then $2^p = 3n - 1$ and $p^2 = 3m + 1$, so that $2^p + p^2$ is always a multiple of 3 for $p > 3$.

M8. In the figure below the oblique lines each have a length of $(20\sqrt{2})/3 = 9.42809$ feet. If these lines are interpreted as the diagonals of the planks then the planks can be marginally longer than $L = \sqrt{791}/3 = 9.3749074$ feet.



M9. Assume that three vertices do fall on grid points. The triangle they form will always include a vertex with an angle of 36 degrees as shown below.



Place that vertex at the origin and suppose the other two vertices of the triangle are at grid points (a, b) and (c, d). Then

$$\cos(36°) = \frac{1+\sqrt{5}}{4} = \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}} \to 3+\sqrt{5} = \frac{8(ac+bd)^2}{(a^2+b^2)(c^2+d^2)}$$

But $\sqrt{5}$ is irrational and therefore cannot be the ratio of two integers. This contradiction proves that three vertices of a regular pentagon cannot lie on grid points.

M10. (a) Weights of 1, 3, 9, and 27 do the job.

(b) Weights of 2, 6, and 18 do the job. Note that a 1-pound package is the only one that won't balance or lift a 2-pound weight.

M11. The longest musical composition without three consecutive repetitions of any sequence are AABABAABABAABAAB and its reverse.

M12. (a) For 5 buttons there are the following types of combinations where S is a single button pushed and P is a pair of buttons pushed: S, 2S, 3S, 4S, 5S, P, PS, P2S, P3S, 2P, 2PS. Corresponding to each of these types the number of combinations is

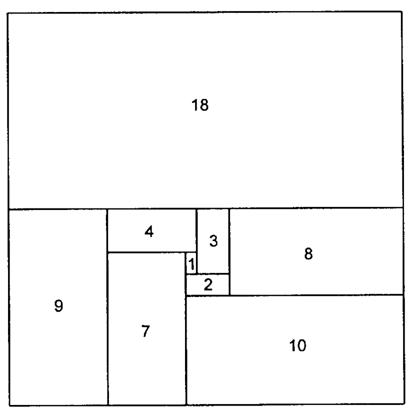
$$\frac{5!}{4!}+\frac{5!}{3!}+\frac{5!}{2!}+\frac{5!}{1!}+\frac{5!}{0!}+\frac{5!}{2\cdot 3!}+\frac{2\cdot 5!}{2\cdot 2!}+$$
$$\frac{3\cdot 5!}{2\cdot 1!}+\frac{4\cdot 5!}{2\cdot 0!}+\frac{5!}{2\cdot 2!}+\frac{3\cdot 5!}{2\cdot 2!} \quad = \quad 935$$

(b) For 6 buttons the types are: S, 2S, 3S, 4S, 5S, 6S, P, PS, P2S, P3S, P4S, 2P, 2PS, 2P2S, and 3P. The number of combinations is

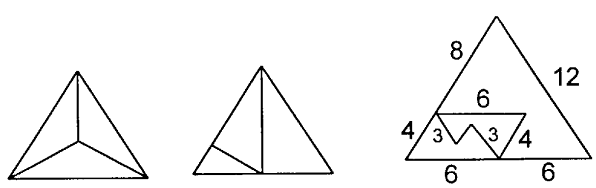
$$\frac{6!}{5!}+\frac{6!}{4!}+\frac{6!}{3!}+\frac{6!}{2!}+\frac{6!}{1!}+\frac{6!}{0!}+\frac{6!}{2\cdot 4!}+\frac{2\cdot 6!}{2\cdot 3!}+\frac{3\cdot 6!}{2\cdot 2!}+$$
$$\frac{4\cdot 6!}{2\cdot 1!}+\frac{5\cdot 6!}{2\cdot 0!}+\frac{6!}{2\cdot 2\cdot 2}+\frac{3\cdot 6!}{2\cdot 2}+\frac{6\cdot 6!}{2\cdot 2}+\frac{6!}{2\cdot 2\cdot 2} = 7671.$$

M13. 1,123,465,789.

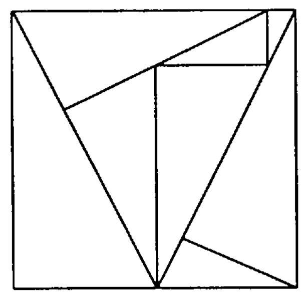
M14. A nine-rectangle solution is shown below. Each number is the length of the shorter side of the rectangle.



M15. Shown below.



M16. An eight-triangle solution is shown below.



M17. (1) The three possibilities are that you have 5, 7 or 11 on your forehead.

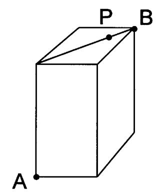
(2) If you have a 5, then person A with a 7 sees (5, 5) and concludes that he must have 3 or 7. But if he has 3 then he reasons that person **B** sees (5, 3) and would know his number is 5 or 3. **B** can eliminate 3 because anyone seeing (3, 3) would immediately know he had 5. Since **B** doesn't know his number, A would conclude that he has a 7. Since A doesn't draw this conclusion you know you don't have a 5.

(3) If you have a 7 then person **B** with a 5 sees (7, 7) and concludes he has 3 or 5. But if he has 3 then he reasons that person A with 7 sees (7, 3) and would know his number is 7. Since A doesn't know his number, then **B** would conclude he has a 5. Since **B** doesn't draw this conclusion you know you don't have a 7. Therefore you have 11 on your forehead.

M18. (a) During the nth minute the band is $100n$ long and the snail travels a fraction of the band equal to $1/100n$. The snail eventually arrives at the far end when $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{t} = 100$. That happens for $t = 1.509269 \times 10^{43}$ minutes.

(b) When the snail has traversed 99 % of the band, the 100-foot stretch causes the far end to move 1 foot farther from the snail. This 1 foot is overcome by the snail's normal progress during that minute. This happens when $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{u} = 99$, or $u = 5.5522899 \times 10^{42} = t/e$.

M19. (a) Unfold the box and examine shortest path distances from A to various points. The farthest is at point **P** one quarter of the way along the diagonal on the 1×1 face from **B**.

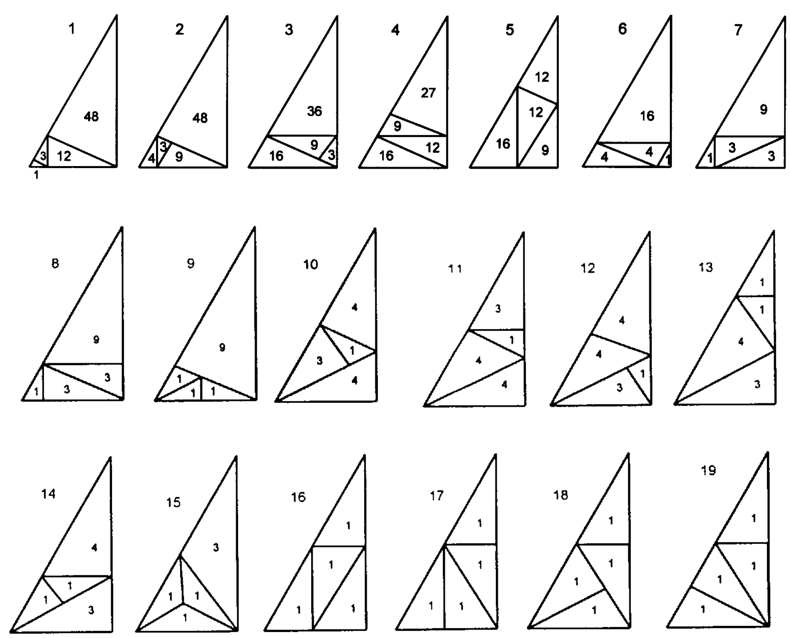


(b) The two points are along the diagonal near P and its opposite point on the 1×1 face that includes A. The points are $(\sqrt{3} - 1)/2$ of the way along each diagonal for a distance of 3.0119....

M20. There are 7 Good Eggs. There are 5 subjects with 15 marks possible in each. The scores for the Good Eggs are (15, 14, 13, 12, 6), (15, 14, 12, 11, 7), (15, 14, 13, 10, 8), (15, 14, 12, 11, 8), (15, 14, 12, 10, 9), (15, 13, 12, 11, 9) and (14, 13, 12, 11, 10). Humpty Dumpty got a 10 in arithmetic.

Solutions to Hard Problems

H1. Shown below. The number in each triangle indicates its area relative to the other triangles in the figure.



H2. Imagine a regular polygon of n sides after Sir George's trip. Its area is $A = nsh/2$, where s is the side length and h is the distance from the center of the polygon to the center of a side: $(2h)/s = \cot(180/n)$. If Sir George travels at speed v then $s/v = (1440 - n)/n$ where 1440 is the number of minutes in a day. To maximize the area is to find the n that maximizes

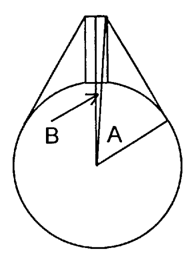
$$A = v^2 \cot(\frac{180}{n})(1440 - n)^2/(4n).$$

The area is largest for $n = 17$, with $A = 159{,}300.1968\ v^2$.

H3. Three equally spaced punches in a straight line do the job if the square of the spacing is irrational.

H4. Let w = width of the building; h = height of the building; r = the earth's radius; and d = stretch in the band due to the building. Then

$$2\tan B = \frac{w}{r+h};$$

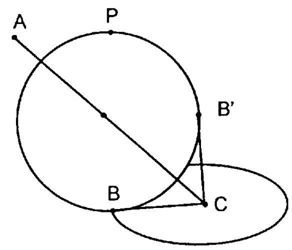


$$r \tan A = \sqrt{2rh + h^2 + \frac{w^2}{4}};$$

$$d = 2r\tan A + w - 2r(A + B).$$

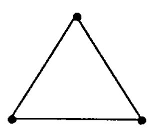
These equations can be iterated to produce $h = 85.763515\ldots$ ft.

H5. Consider the ball rotating about the axis *AC* without slipping, as shown below. The cone, *BCB'* rolls along the plane. The distance from *B* to the axis *AC* is $\sqrt{0.5}$ of the distance *BC*. Thus *B* will be in contact with the horizontal plane when $\sqrt{0.5}$ rotations about the axis *AC* have occurred. Point *P* will return to the top again as well. For a full rotation of the ball on the circle the point *P* will execute $\sqrt{2}$ rotations about the axis. Relative to the center point of the ball, the point *P* will have coordinates $P = -r\sin\theta/\sqrt{2}, r(\cos\theta - 1)/2, r(\cos\theta + 1)/2$, where $\theta = \sqrt{8}\pi$. For $r = 1, P = (-0.3629497, -0.9291081, 0.0708919)$; its initial position was (0, 0, 1).

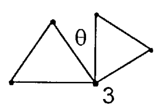


H6. The solutions are shown below.

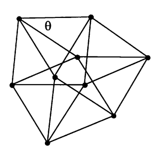
(a)



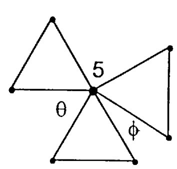
(b) $\theta \neq 0°, 60°, 180°, 240°, \text{or } 300°.$



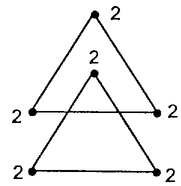
(c) $\theta \neq 0°, 60°, \text{or} 180°.$



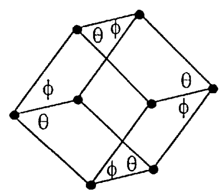
(d) θ, ϕ and $\theta + \phi + 60° \neq 0°, 60°, 180°, 240°,$ or $300°.$



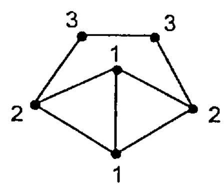
(e)



(f) θ, ϕ and $\theta + \phi \neq 0°, 60°$, or $120°$.
$\cos(2\phi) + \cos(2\theta) \neq 2\cos\phi + 2\cos(\theta) - \cos(\theta - \phi)$.



(g)



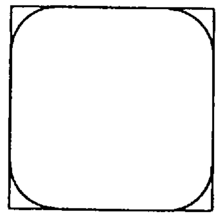
H7. (a) $.1^{-.5} = 3.162276\ldots$

(b) $\dfrac{\sqrt{\sqrt{\sqrt{\sqrt{.9}}}}}{\sqrt{.1}} = 3.141522\ldots.$

(c) Define $a(n) = \sqrt{\cdots\sqrt{a}\cdots}$ where there are n square roots. Clearly $c = a(n) - b(m)$ can be made arbitrarily close to 0. For the right k, $c(k)$ can be put in the interval $0.25 - 0.5$. Pick a and b as any digits other than 0. There are an infinite number of choices of m and n followed by the appropriate k to make an infinite density of $c(k)$'s in the $0.25 - 0.5$ interval, which contains $1/\pi$.

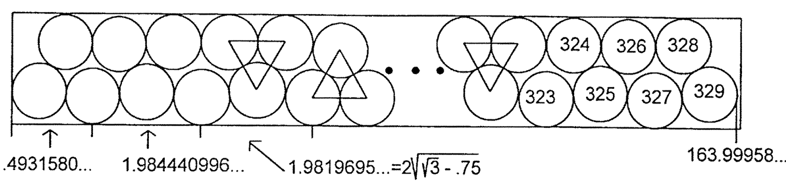
H8. (a) Imagine an expanding soap bubble in the unit square. Its form will take on the shape of four quarter circles of radius r at the corners as shown in the figure. The area of such a shape is $A = 1 + (\pi - 4)r^2$.

The perimeter of the shape is $P = 2\pi r + 4 - 8r$. The ratio A/P takes on a maximum when $r = 1/(2 + \sqrt{\pi}) = 0.265079\ldots$. The maximum ratio $A/P = r = 0.265079\ldots$. Note that the ratio A/P is 0.25 for the entire square or a circle inscribed in the square.

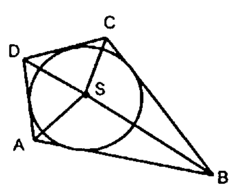


(b) The corresponding problem of maximizing the ratio of volume to surface area within the unit cube remains unsolved.

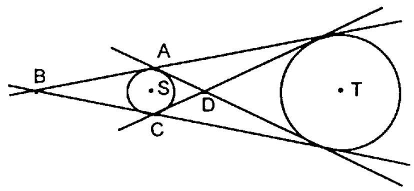
H9. The circles need to be packed as shown below. For $n = 164$ there is just enough room for 329 circles. There are 7 circles on each end with 105 sets of 3 circles in the middle. The smallest rectangle found to date containing 329 circles has 13 circles on each side of 101 sets of 3 and measures $2 \times 163.9973967\ldots$.



H10. A view of the pyramid looking down from *S* perpendicular to the plane *ABCD* will look like the figure below.



The point *S* may be raised above the plane to give unit altitudes if the inscribing circle has radius < 1. Now consider two such circles of different sizes as shown. Each has a radius < 1. They define the four tangent lines drawn to produce *ABCD* as shown. Raise *S* and *T* appropriately above the plane so that all altitudes are unit. Clearly ST is not perpendicular to the plane.



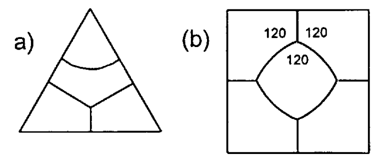
H11. Let $P(A-B)$ be the probability of the server winning the game when the server has A points and the receiver has B points. Let $p =$ the probability of the server winning a point, and let $q = 1 - p$.
$P(40-40) = pP(40-30) + qP(30-40)$;
$P(40-30) = p + qP(40, 40)$;
$P(30-40) = pP(40-40)$.
This leads to
$P(40-40) = \frac{p^2}{p^2+q^2}$;
$P(40-30) = p + \frac{p^2 q}{p^2+q^2}$.
Similarly one derives
$P(40-15) = p + pq + \frac{p^2 q^2}{p^2+q^2}$;
$P(30-15) = p^2(1+q) + \frac{p^2 q(pq+1)}{p^2+q^2}$;
$P(0-0) = \frac{p^4(1-16q^4)}{p^4-q^4}$.
At a score of n games to n ($n = 0$ to 5), $P(0-0) > P(30-15)$ when $8p^2 - 4p > 3$, which leads to $p > 0.911437827$. At n games to n, $P(0-0) > P(40-30)$ when $8p^3 - 4p^2 - 2p > 1$, which leads to $p > 0.919643377607$.

H12. A computer program was written to calculate the probabilities shown in the table below. My uncle is expected to take the longest time to dress on Saturday and on Friday he is least likely to get a pair from the first three socks chosen.

Day	Average Selections Required	Pairing probability
Monday	1.2069	0.8286
Tuesday	1.2270	0.8163
Wednesday	1.2511	0.8027
Thursday	1.2801	0.7892
Friday	1.3074	0.7805
Saturday	1.3586	0 .7849

H13. We look for cases where (1) the oldest child is under 20, (2) the younger two children have different ages, and (3) there is a product and sum that give rise to ambiguity for the ages both this year and two years ago. There is only one set of ages that accomplishies this: (5, 6, 16). The product and sum could be achieved by (4, 8, 15), which must have been guessed by Smith. The product and sum two years ago could be achieved by (2, 7, 12), which must have been guessed by Jones two years ago.

H14. (a) Length = 1.342181807 as shown.
(b) Length = 2.50211293 as shown.
(c) Unknown.

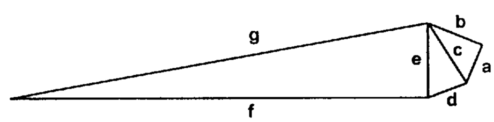


H15. Using a multiprecision continued fraction program, we get

(a) $a = 7.187824 \times 10^{288}$, $b = 2.69442527 \times 10^{288}$, $g = 0.010659$.
(b) $c = 4.744902 \times 10^{154}$, $d = 1.51034933 \times 10^{154}$, $g = 0.007190$.

H16. (a) $a =$ prime, $b = (a^2 - 1)/2$, $c = (a^2 + 1)/2 =$ prime, $d = (c^2 - 1)/2$, $e = (c^2 + 1)/2 =$ prime.

Solutions occur for $a = 3, 11, 19, 59, 271, 349, 521, 929, 1031, 1051, 1171, \ldots$.
(b) $a = 271$; $b = 36,720$; $c = 36,721$; $d = 674,215,920$; $e = d + 1$; $f = 227,283,554,064,939,120$; $g = f + 1$.



H17. (a) 45 years.
(b) 605 years.
(c) 17,042,641,444 years.
(d) Yes; a=10,093,613,546,512,321 is the first of 5 in a row.
$a = 49p1, a + 1 = 2 * p * p, a + 2 = 9 * p2, a + 3 = 4 * p3, a + 4 = 25 * p4.$

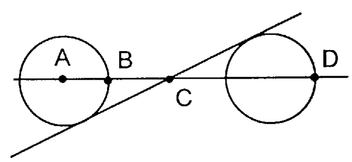
Six in a row is impossible because three consecutive even special years cannot occur.

H18. The venture should be undertaken since the volume in cubic spandrals can be determined using calculus to be

$$V = \frac{4\pi}{3}(1.331)^{2/3}(60)^3 = 1,094,782.208.$$

It is noteworthy that the volume doesn't depend on the polar or equilateral radii of Alpha Lyra IV.

H19. The figure below shows a cross-section of the torus.



Let $r = AB$ and $R = CD$. There are three classes of channels that can be dug on its surface.

(a) It is clear that many channels with radius r are possible.
(b) Channels with radii between $R - 2r$ and R are possible.
(c) A less well-known third type of circular channel with radius $R - r$ is possible, which is the intersection of the plane shown by the slanting line and the torus.

From the descriptions of the first two students they must have dug channels of type (b). The third and fourth students must have explored channels of type (a) and (c) in some order. One case gives $R - r = 25$ and $r = 30$; the other case gives $R - r = 30$ and $r = 25$. The first case is not possible because $R - 2r < 0$. Thus the second case applies and $2\pi R = 110\pi$.

H20. (1) Define

$$I(n, x) = \sum_{k=0}^{n} \frac{(k-n)^k e^{(k-n)x}}{k!} = \sum_{k=0}^{n} \frac{1}{k!}\frac{d^k}{dx^k} e^{(k-n)x}$$

Clearly $F(n) = 2n + \frac{2}{3} - I(n, -1)$.

(2) Since $e^{(k-n)z}$ is an entire function in the complex plane, it follows from Cauchy's Theorem that

$$F(n) = 2n + \frac{2}{3} - \sum_{k=0}^{n} \frac{1}{2\pi i} \int_C dz \frac{e^{(k-n)z}}{(1+z)^{k+1}},$$

where C must enclose $z = -1$ and will be taken to be $|z| = 2$.

(3) The summation over k can be carried out explicitly to give

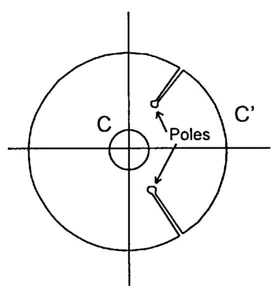
$$F(n) = 2n + \frac{2}{3} + \frac{1}{2\pi i} \int_C dz \frac{e^{-(n+1)z}}{1-(1+z)e^{-z}} - \frac{1}{2\pi i} \int_C \frac{dz}{[1-(1+z)e^{-z}](1+z)^{n+1}}$$

(4) Define $q(z) = 1/[1-(1+z)e^{-z}]$. The only pole of $q(z)e^{-(n+1)z}$ within C is a double pole at $z = 0$. Thus $z^2 q(z) e^{-(n+1)z}$ is regular within C, and by Cauchy's Theorem we have

$$\frac{1}{2\pi i} \int_C dz q(z) e^{-(n+1)z} = \frac{d}{dz} \left[z^2 q(z) e^{-(n+1)z} \right]_{z=0} = -(2n + \frac{2}{3})$$

Thus

$$F(n) = -\frac{1}{2\pi i} \int_C \frac{dz q(z)}{(1+z)^{n+1}}$$



To evaluate $F(n)$, we expand and deform C to a contour C' like that in the figure, which still avoids enclosing the poles of $q(z)$. As C' gets arbitrarily large it still encloses the same poles as C. Thus

$$\begin{aligned} F(n) &= -\frac{1}{2\pi i} \int_{C'} \frac{dz q(z)}{(1+z)^{n+1}} \\ &= -\frac{1}{2\pi i} \int_{|z|=R} \frac{dz q(z)}{(1+z)^{n+1}} + \frac{1}{2\pi i} \sum_{k=1}^{k=\infty} \int_{C_k} \frac{dz q(z)}{(1+z)^{n+1}}, \end{aligned}$$

where C_k is the counterclockwise path about the k^{th} pole of $q(z)$, not including its pole at $z = 0$.
(5) On $|z| = R$ for sufficiently large R, $q(z)$ is bounded and the integral on the path tends to zero as R tends to infinity. Thus $F(n)$ is the sum of integrals on small circle paths about the poles of $q(z)$. A pole of $q(z)$ occurs at $1 + z = e^z$, where $z = x + iy$. The resulting equations in x and y are $\sin y = ye^{(1-y\cot y)}$ and $x = \ln(y/\sin y)$. Since the equation in y is even, we can look for poles in the upper half-plane only and reflect each one into the lower half-plane. Call the k^{th} such pole in the upper half-plane $z_k = x_k + iy_k$. The table below gives numerical values of the first few. For larger k define $\phi_k = (2k + 1/2)\pi$. Then y_k approximately equals $\phi_k - [1 + \ln(\phi_k)]/\phi_k$.

k	x_k	y_k
1	2.0888430	7.4614892
2	2.6406814	13.8790560
3	3.0262969	20.2238350
4	3.2916783	26.5432385
5	3.5012690	32.8505482
6	3.6745053	39.1512074
7	3.8221528	45.4473849

(6) The contribution to $F(n)$ from pole k is the limit as z approaches z_k of $(z - z_k)q(z)/(1 + z)^{n+1}$, which reduces to

$$(\cos(n+1)y_k - i\sin(n+1)y_k)\,\frac{(1 + x_k + iy_k)(x_k - iy_k)}{D},$$

where $D = (x_k^2 + y_k^2)e^{(n+1)x_k}$.
Combining the poles in the upper and lower half-planes and some rearrangement finally produces

$$F(n) = \sum_{k=1}^{\infty} A_k e^{-nx_k}\sin(ny_k - \theta_k),$$

where

$$A_k = \frac{-2\sec(y_k + \theta_k)e^{-x_k}y_k}{x_k^2 + y_k^2}$$

$$\tan(y_k + \theta_k) = \frac{x_k^2 + x_k + y_k^2}{y_k}.$$

(a) Since $x_k > 0$ for all k, e^{-nx_k} goes to zero as n goes to infinity.

(b) For $n > 10$, $F(n)$ is dominated by the first pole and its reflected pole. Thus $F(1000)$ is very nearly $A_1 e^{-1000x_1} \sin(1000y_1 - \theta_1)$, giving $F(1000) = -1.14698 \times 10^{-909}$.

(c) From observing the behavior of $\sin(ny_1 - \theta_1)$ one can determine the smallest m where the magnitude of $F(m)$ is less than the magnitude of $F(m+1)$. It is at $m = 800$.

$$F(800) = 5.06175 \times 10^{-728}, F(801) = -5.12298 \times 10^{-728}$$

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