O'Beirne's Hexiamond

Richard K. Guy

Tom O'Beirne is not as celebrated a puzzler as he deserves to be, particularly on this side of the Atlantic. When I began to write this article, I was sure that I would find references to the Hexiamond in Martin Gardner's column, but I haven't found one yet. Nor is it mentioned in O'Beirne's own book [4]. But it does appear in his column in the *New Scientist*. Maybe the only other places where it has appeared in print are Berlekamp et al. [1] and the not very accessible reference Guy [2].

It must have been in 1959 that O'Beirne noticed that, among the shapes that can be formed by adjoining six equilateral triangles, five had reflexive symmetry, while seven did not. Martin [3] uses O'Beirne's names for the shapes but does not distinguish between reflections so his problems only involve 12 shapes. If we count reflections as different, then there are 19 shapes (Figure 1). One of these is the regular hexagon, which can be surrounded by six more hexagons, and then by twelve more, giving a figure (Figure 2) having the same total area as the 19 shapes. Question: Will the 19 shapes cover the figure? It took O'Beirne some months to discover that the answer to the question is "Yes!" Figure 3 was discovered in November 1959.

O'Beirne thought that the result would be more pleasing if the Hexagon were in the center, and in January 1960 he found the solution shown in Figure 6. In the interim he had found solutions with the Hexagon in two other of its seven possible positions (Figures 4 and 5).

It was soon after this that O'Beirne visited the Guy family in London. He showed us many remarkable puzzles, but the one that grabbed us the most was the Hexiamond, and several copies had to be manufactured, since everyone wanted to try it at once. No one went to bed for about 48 hours. The next solution in my collection is Figure 7 by Mike Guy (March 1960).

We became adept at finding new solutions based on the old. Remove pieces $\bar{1}$, $\bar{3}$, 6, and 8 from Figure 3 and replace them in a different way. Or try using 0, 1, 3, 5, 6, and $\bar{8}$. Rearrange pieces 1, $\bar{1}$, 3, 5, and $\bar{8}$ in Figure 5; and $\bar{3}$, $\bar{6}$, 7, and $\bar{9}$ in Figure 7. It soon became necessary to devise a classification scheme, since it was not easy to decide whether a solution was new or not.

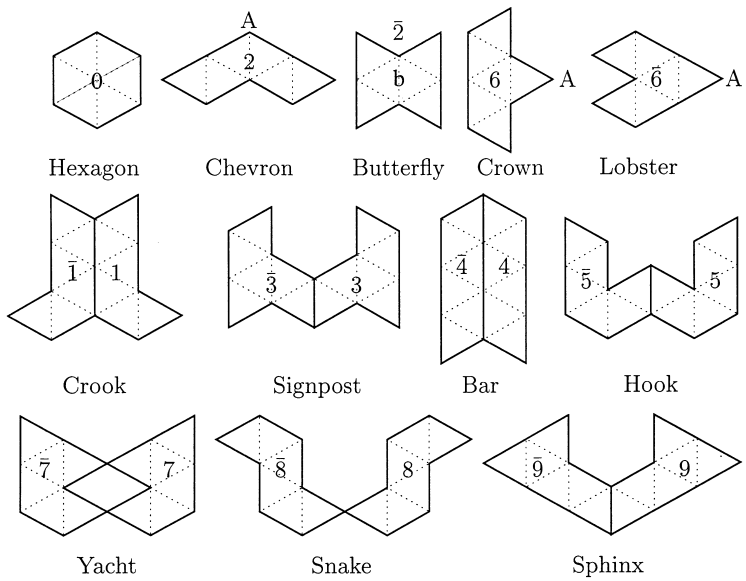


Figure 1. The nineteen Hexiamond pieces. The symmetries of the Hexagon are those of the dihedral group D6. The Butterfly has the same symmetries as a rectangle: Its position is described by that of its body. The Chevron, Crown, and Lobster each have a single symmetry of reflection, and each is positioned by its apex. Pieces 4 and 8 have only rotational symmetry and exist in enantiomorphous pairs, as do pieces 1, 3, 5, 7, and 9, which have no symmetry.

O'Beirne had already suggested a numbering of the pieces from 1 to 19. By subtracting 10 from the Hexagon, and subtracting each of the labels 11 to 19 from 20 we arrive at the labeling in Figure 1. Notice that 0 has the greatest symmetry, the multiples of 4 have rotational symmetry through 180 degrees, and the other even numbers have reflective symmetries. Pieces with odd-numbered labels have no symmetry and, together with those that have rotational symmetry only, exist in enantiomorphous pairs. We don't know how O'Beirne decided whether to give a piece a number less than or greater than 10, but our mnemonics are as follows. Negative (bar sinister?) labels are given to the Bar, $\bar{4}$, a parallelogram drawn in the opposite way from the usual textbook fashion, to the Snake, $\bar{8}$, which appears to be turning to the left, to the Crook, $\bar{1}$, with its hook on the left, to the Signpost, $\bar{3}$, with its pointer on the left, to the Hook, $\bar{5}$, with its handle on the left, to the Yacht, $\bar{7}$, sailing to the left, and the Sphinx, $\bar{9}$, whose head is on the left.

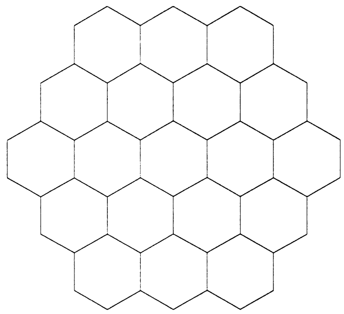


Figure 2. Hexiamond board.

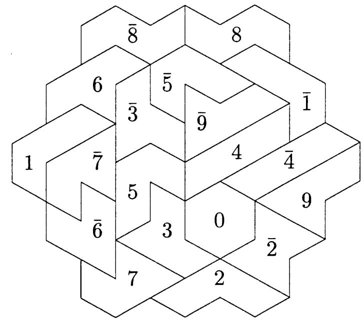


Figure 3. O'Beirne's first solution (November 30, 1959).

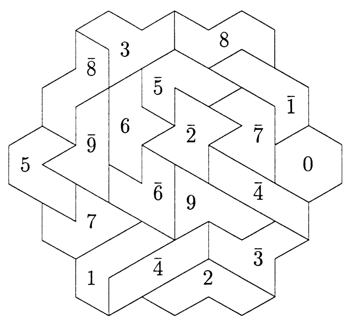


Figure 4. $hA_3f_4K_2l_5$ 60-01-05

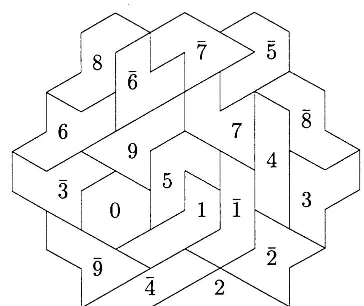


Figure 5. $hD_8p_4F_xd_7$ 60-01-10

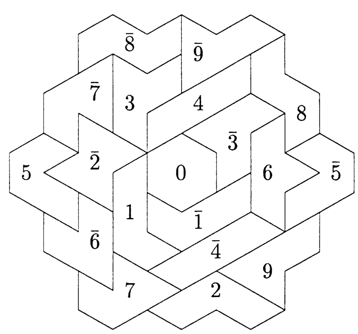


Figure 6. $hGh_xU_3a_5$ 60-01-21

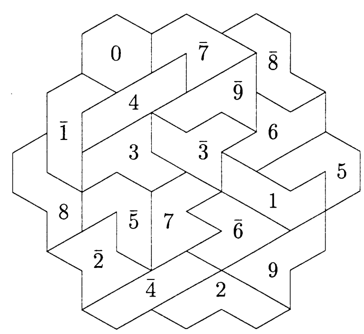


Figure 7. $hA_\varepsilon i_8B_\varepsilon i_3$ 60-03-27

We classify solutions with a string of five labels, giving the positions of the Chevron, Hexagon, Butterfly, Crown, and Lobster. This doesn't completely describe the solution, but gives enough detail to enable comparisons to be made quickly. We won't count a solution as different if it is just a rotation or a reflection, so first rotate the board so that the Chevron, piece number 2, is pointing upward. Then, if the Chevron is on the left-hand side of the board, reflect the board left to right in order to bring it onto the right half.

Read off the lowercase letter at the apex, *A*, of the Chevron from Figure 8. Note that the letters *a* and *n* are missing; it is possible to place the Chevron in such positions, but it's clear that they can't occur in a solution. More than three-quarters of the positions found so far have the Chevron in position *h*. (The exact figure is 89.5%.)

Next use Figure 9 to describe the position of the Hexagon, piece number 0. This will be a capital letter, *A* to *G*, depending on how far it is from the center. If the Chevron is central (positions *b* to *g* in Figure 8), reflect the board if necessary to bring the Hexagon into the right half. Then append a subscript $0, 1, \ldots, 9, x, \varepsilon$ indicating its position on the clock. *A* and *E* have only odd subscripts, *B, D, F* only even ones, and *G*, the center, is unique and requires no subscript. The subscripts on *C* are about $\frac{1}{2}$ more than their clock hour.

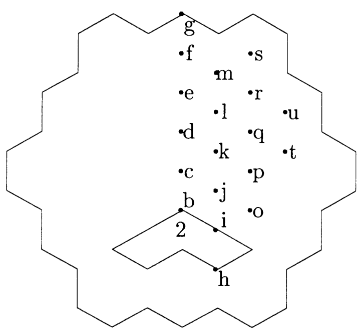


Figure 8. Coding the Chevron.

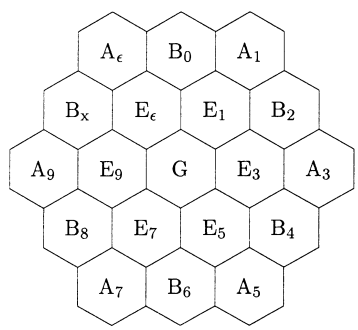


Figure 9. Coding the Hexagon.

The position of the Butterfly, piece number $\bar{2}$, is that of its "body," the edge that bisects it. This is indicated by a lowercase letter, shown in Figure 10, together with an even subscript, 0, 2, 4, 6, 8, or x, the side of the board it is nearest to. The subscripts *a, e, l, s,* and *t* are omitted, since placing the Butterfly there does not allow a solution to be completed; *f* and *m* are equidistant from opposite sides of the board, and are given only 0, 2 or 4 for a subscript. If both Chevron and Hexagon are symmetrically placed, reflect the board if necessary to bring the Butterfly into the right half. If all three are symmetrically placed, as in the first solution in Figure 14, reflect

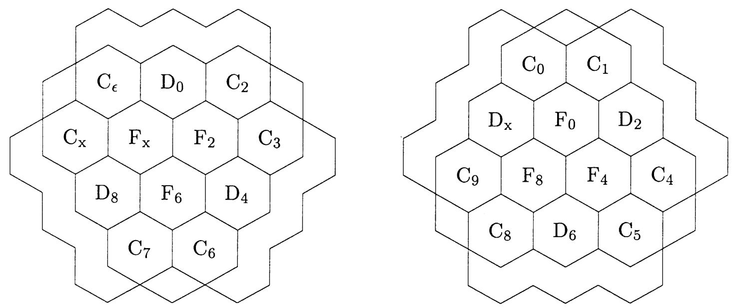


Figure 9. Continued: Coding the Hexagon's position.

if necessary to make the apex of the Crown point toward the right half of the board.

The position of the Crown, piece number 6, is given by the capital letter at its apex, *A*, in Figure 11. The subscript is even or odd according to whether the Crown is to the left or right of the axis of symmetry of the board in the direction in which its own axis of symmetry is pointed. Positions *S* and *T* are symmetrical and carry even subscripts. Positions *C, F, M, Q, R* are omitted, as they don't allow legal solutions.

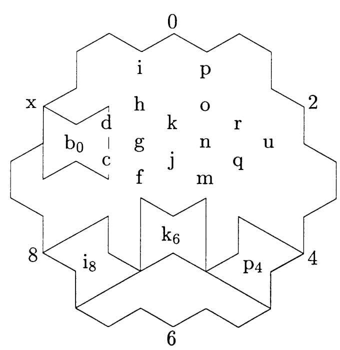


Figure 10. Coding the Butterfly.

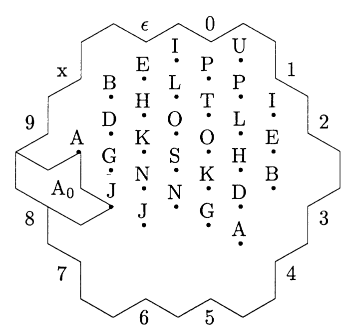


Figure 11. Coding the Crown.

The Lobster, piece number $\bar{6}$, is located by a lowercase letter in Figure 12. Except for *a*, these are in the same positions as the capital letters used for the Crown. Again the subscript is even or odd according to whether the piece is to the left or right of the axis of symmetry of the board in the direction its tail is pointing. Positions *s, t,* and *u* are symmetrical and carry only even subscripts; *c, g;* and *r* don't lead to legal solutions.

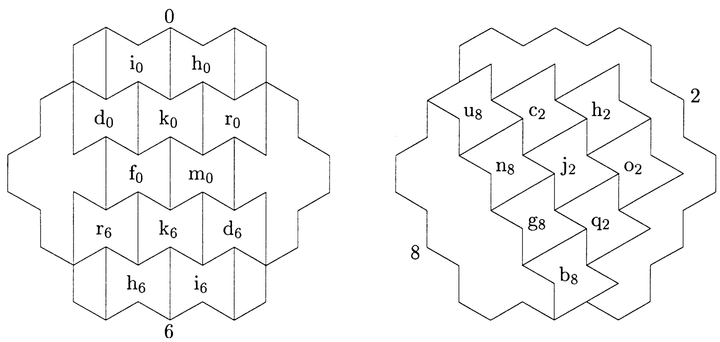


Figure 10. Continued: Coding the Butterfly's position.

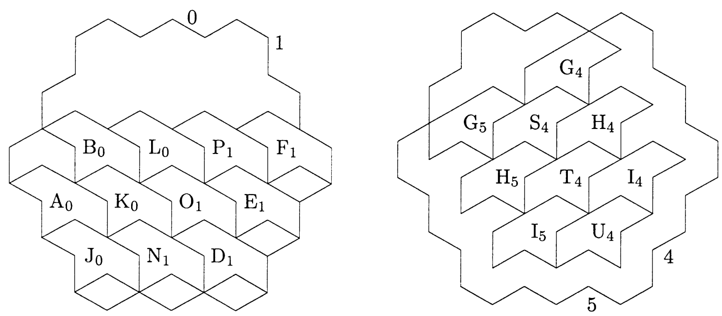


Figure 11. Continued: Coding the Crown's position.

Now that you know how to classify solutions, note that O'Beirne's first solution (Figure 3) is of type $hE_5p_4J_5a_5$. Remember that the code doesn't specify the solution completely. There are at least ten solutions of class $hA_9p_4B_0f_0$, for example. (Can you find a larger class?)

How many solutions are there? A wild guess, based on how rarely duplicates appear, is about 50,000. There are already more than 4200 in the collection, which will be deposited in the Strens Collection in the Library at the University of Calgary. It isn't very easy to give a good upper bound. Coloring arguments don't seem to lead to much restriction, but perhaps some reader will be more perspicacious.

There are 508 essentially different relative positions for the Chevron and Hexagon that have not been proved to be impossible, although some of these turned out to be so. We have found 247 of these cases (no fewer than twenty-six bit the dust during the writing of this article). With the Chevron in positions *g, h,* and *o,* there are respectively 21, 39, and 1 legal

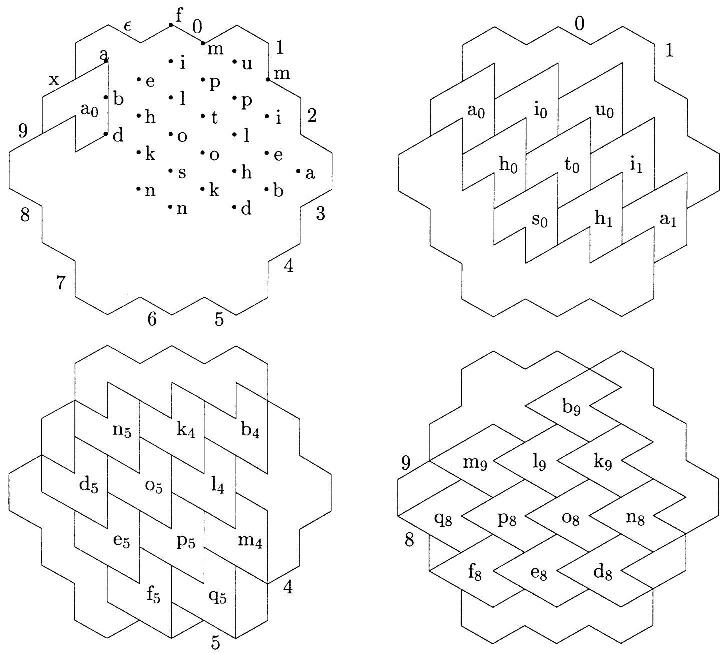


Figure 12. Coding the position of the Lobster.

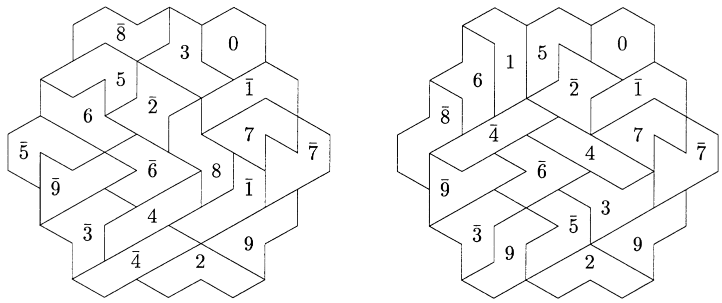
positions for the Hexagon, and solutions are known in all of these cases. As we go to press, Marc Paulhus has established that there are just

$$13 + 17 + 16 + 14 + 18 + 21 + 39 + 29 + 27 + 30 \\ + 28 + 30 + 1 + 23 + 33 + 7 + 33 + 22 + 32 = 433$$

different relative positions for the Chevron and Hexagon that yield solutions.

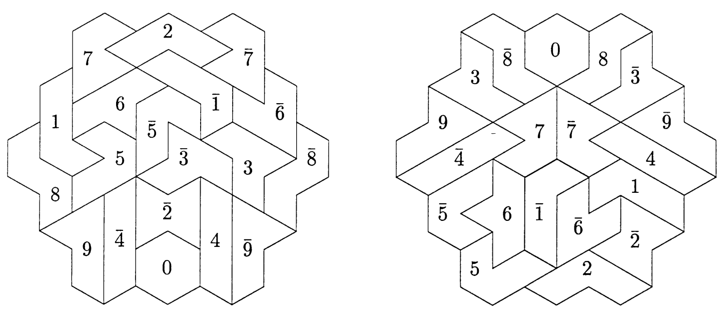
In 1966 Bert Buckley, then a graduate student at the University of Calgary, suggested looking for solutions on a machine. I didn't think he would be successful, but after a few months of intermittent CPU time on an IBM 1620 he found half a dozen or so solutions from which it was possible to deduce another fifty by hand. Figure 13 shows two of Bert Buckley's machine-made solutions.

With such a plethora of solutions, the discerning solver will soon wish to specialize. For example, how symmetrical a solution can you get? Figure 14 shows two solutions, the first found by John Conway and Mike Guy in 1963, the second by the present writer, each with as many as 11 of the 19 pieces symmetrically placed.



$hA_1n_xB_0o_3$ 66-05-30 $hA_1q_xF_9o_3$ 66-08-15

Figure 13. Solutions found by machine by Bert Buckley.



$gB_6k_6N_5a_4$ $hB_0p_4I_8i_6$

Figure 14.

Notice that there are two different kinds of axes of symmetry. All edges of the pieces lie in one of three different directions, at angles of 60 degrees to one another. We have always drawn the hexiamond with one of these directions vertical, but you may prefer to put one of them horizontal; Figure 15 shows eight solutions, each with 11 symmetrically placed pieces, found by John Conway in 1964. To deduce the others from the single diagram shown, rearrange pieces $2\bar{2}9\bar{9}$ or $3\bar{3}8\bar{8}$ and note that the hexagon formed by $01\bar{1}6\bar{6}$ has the symmetries of a rectangle.

Conway showed that you can't have more than 15, respectively 13, pieces placed symmetrically with respect to the two kinds of axis. It seems unlikely that anyone can beat 11.

Some solutions come in large groups. For example, in Figure 14, rearrange pieces $\bar{2}3\bar{3}$, or $\bar{1}\bar{3}5\bar{5}6$, after which you can swap $0\bar{2}$ with $\bar{3}\bar{5}$. You

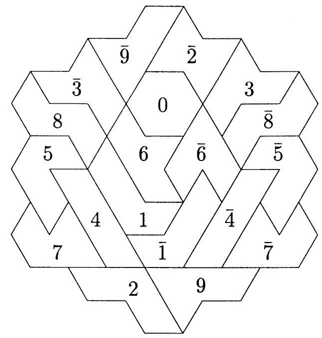


Figure 15. $hE_\varepsilon i_0 O_0 l_\varepsilon$ The most symmetrical solution?

can rotate 0$\bar{2}\bar{4}$ ($\bar{3}\bar{4}\bar{5}$) or 0$\bar{2}$4 ($\bar{3}$4$\bar{5}$) and produce two "keystones," which can be swapped, and so on.... Figure 16 shows a solution displaying three keystones that can be permuted to give other solutions. Figure 17, found by Mike Guy, shows that keystones (4$\bar{4}\bar{9}$) can occur internally and don't have to fit into a corner.

Figure 18 has the Hexagon in position *G* and the other pieces forming three congruent sets. Figure 19 is one of at least eight examples that have six pieces meeting at a point, here $\bar{3}$5$\bar{5}\bar{6}$7$\bar{7}$. Readers will no doubt discover other curiosities and find their own favorites.

In an earlier draft I wrote that with modern search techniques and computing equipment and some man–machine interaction it has probably become feasible to find all the solutions of the Hexiamond. In May 1996

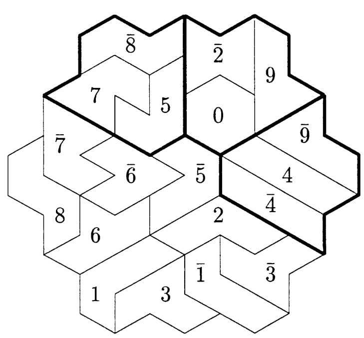


Figure 16. $jE_1 p_0 B_x s_2$ has three keystones.

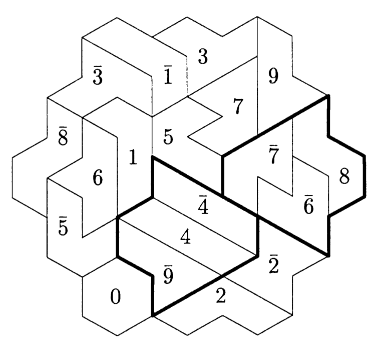


Figure 17. $hA_7 p_4 U_8 f_4$ has an internal keystone.

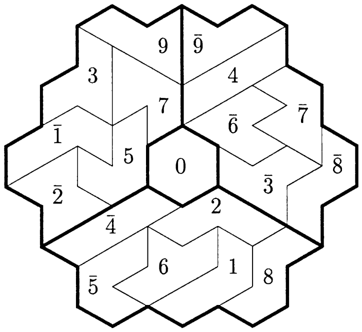


Figure 18. $jGp_8G_xk_9$ (3-symmetry).

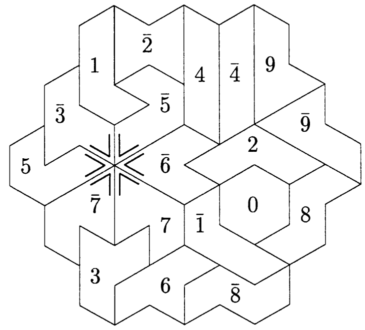


Figure 19. $qD_4i_0F_5t_8$
Six pieces meet at a point.

Marc Paulhus wrote a program that used only a few days of computer time to find all the solutions.

Their numbers, classified according to the position of the Chevron, are

b	c	d	e	f	g	h	i	j	k
130	195	533	193	377	2214	111,460	584	985	885

l	m	o	p	q	r	s	t	u	Total
637	914	749	498	1238	31	1537	264	1094	124,518

and, by the Hexagon's position

A 75,489 *B* 15,717 *C* 6675 *D* 7549 *E* 11,447 *F* 5727 *G* 1914

I don't think that this need take the fun out of one of the best two-dimensional puzzles ever invented. On the contrary, for those who prefer their puzzles to have just one answer, there are no fewer than 40 relative positions of the Chevron and Hexagon that determine such a unique solution:

bC_1	bC_3	bD_0	bE_3	cB_2	cC_1	cC_2	cC_5	cC_6	cD_6
cF_0	eC_3	eC_5	fC_6	fF_2	iC_1	iC_x	jD_0	kC_8	kC_9
lD_x	lF_6	mC_5	pF_8	pF_x	pG	qC_0	qF_x	rB_8	sC_x
sC_ε	tB_8	tB_x	tC_1	tC_x	tE_1	tE_7	tF_0	tF_6	tF_x

How long will it take you to find them all without peeking at Marc's database [5]?

After writing a first draft of this article, I wrote to Kate Jones, president of Kadon Enterprises and discovered that she markets a set of "iamonds" under the trademark "Iamond Ring." For this purpose the iamonds are not counted as different if they are reflections of one another, so, if you want to use them to make O'Beirne's hexiamond puzzle, you must get two sets. The numbers of iamonds, diamonds, triamonds, . . . , are given in the following table, where the last line counts reflections as different:

Order	1	2	3	4	5	6	7
# of iamonds	1	1	1	3	4	12	24
with reflections	1	1	1	4	6	19	44

The Iamond Ring is a beautiful arrangement, discovered by Kate herself. Figure 20, in which each *x*-iamond is labeled $x, 1 \leq x \leq 7$, doesn't do justice to the very appealing colored pieces of the professionally produced puzzle.

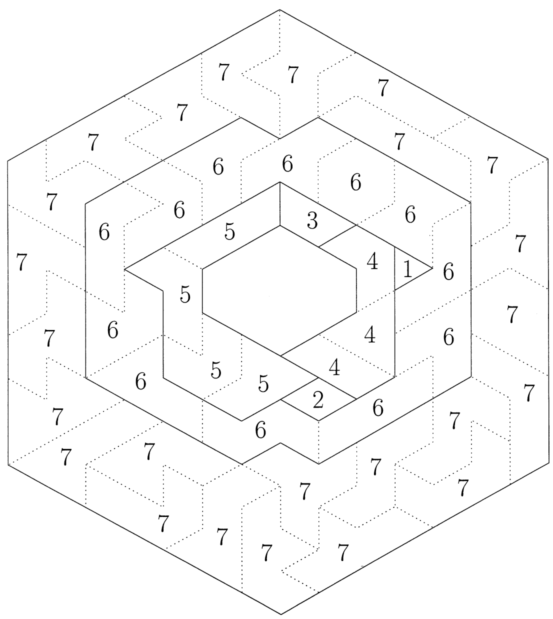


Figure 20. Kate Jones' Iamond Ring.

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays,* Academic Press, London, 1982, pp. 787–788.

[2] Richard K. Guy, Some mathematical recreations I, *Nabla (Bull. Malayan Math. Soc.),* 9 (1960) pp. 97–106; II pp. 144–153; especially pp. 104–106 and 152–153.

[3] George E. Martin, *Polyominoes: A Guide to Puzzles and Problems in Tiling,* Math. Assoc. of America Spectrum Series, 1991, pp. 168–170.

[4] Thomas H. O'Beirne, *Puzzles and Paradoxes,* Oxford University Press, 1965; see Puzzles and Paradoxes No. 44: Pentominoes and hexiamonds, *New Scientist,* 259 (61-11-02) pp. 316–317.

[5] Marc Paulhus, *A database for O'Beirne's Hexiamond,* submitted.