Cube Puzzles

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Binomial Puzzle

A new, rather amusing, combinatorial puzzle can be constructed by acquiring twenty-seven uniform cubes of size $a \times a \times a$ and gluing them together to form the following eight pieces:
One "Smally" of size $a \times a \times a$ (i.e., a single cube).
One "Biggie" of size $b \times b \times b$ (where $b = 2a$).
Three "flats" of size $a \times b \times b$.
Three "longs" of size $a \times a \times b$.
Now color each piece with six colors according to the scheme in Figure 1.

The problem then is to arrange the eight pieces into a cube so that opposite faces have the same color.

This puzzle, if colored as above, has exactly two solutions — each with a different set of three colors. With proper insight it can be solved in a few minutes. Without this insight it typically takes several hours to arrive at a solution if one can be found at all. Before proceeding with this discussion the reader is urged to build a puzzle and attempt its solution.

Most Martin Gardner aficionados will recognize that the eight pieces model the situation represented by the binomial expansion

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \tag{1}$$

Gardner calls such models of mathematical theorems "look-see" proofs. In fact he has recommended that every teacher of algebra construct a set of eight pieces for classroom use. Usually there is an "aha" reaction when students see that a cube can actually be constructed from the pieces. For more advanced students it would be reasonable to ask in how many essentially different ways can the cube be constructed from the eight (uncolored) pieces. This will depend on just what is meant by the words "essentially different" but one interpretation could be to orient the cube to sit in the positive octant

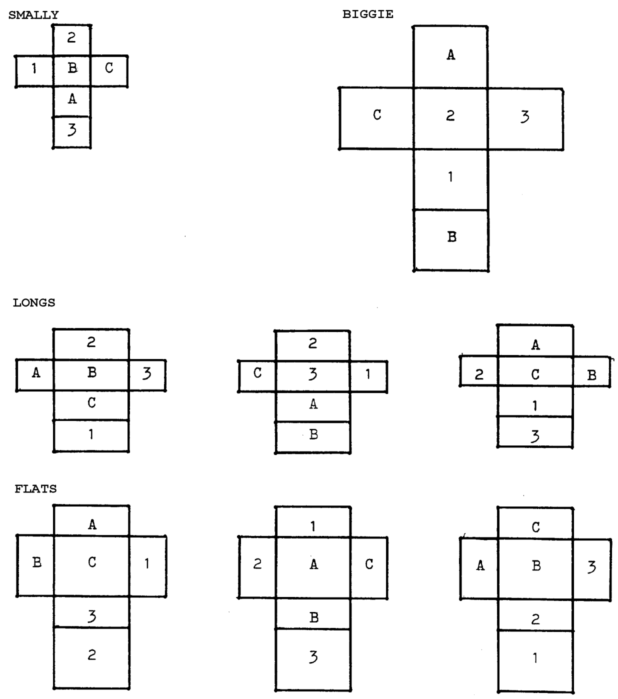

Figure 1. A, B, C, 1, 2, and 3 are any six distinctive colors.

in space with Biggie always occupying the corner (0, 0, 0). There are 93 solutions in this case. If, additionally, each of the 48 faces of the pieces is colored with a unique color, then there are $(93)(8^8)(3!)^2 = 56{,}170{,}119{,}168$ distinct ways of constructing the cube.

The combinatorial puzzle uses only six colors in its construction, and the solver has the additional clue that a face is all of one color; but there still remain a great many cases that are nearly right. Trial and error is not a very fruitful way of trying to solve this puzzle.

Solution Hints. These hints are to be regarded as progressive. That is, after reading a hint, try again to solve the puzzle. If you cannot, proceed to the next hint.

(1) Call the six colors A, B, C, 1, 2, and 3. A, B, and C will turn out to be a solution set. Notice that the eight pieces must form the eight corners of the completed cube — one piece for each corner. Therefore, each piece must have on it a corner with the three

colors A, B, and C in some order. (It is a fact, but not necessary for the solution, that four pieces will have the counterclockwise order A–B–C and the other four the order A–C–B.)

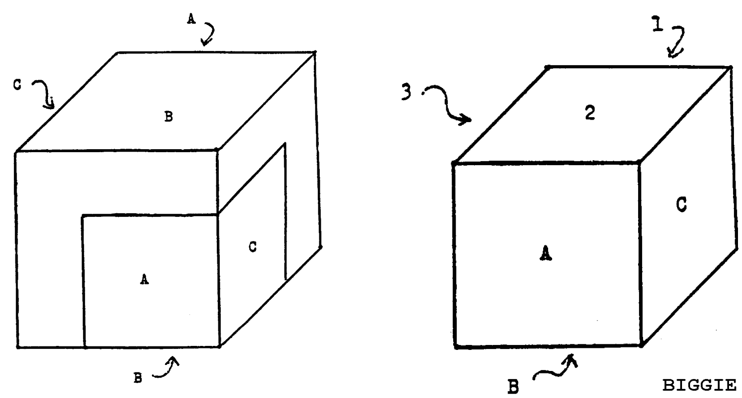

Figure 2.

(2) To determine the colors A, B, and C, take any piece (Biggie is a good choice), and notice that the three pairs of colors A–1, B–2, and C–3 oppose each other. This means that A and 1 cannot appear together in a solution. Likewise B and 2 cannot, nor can C and 3. Of the 20 possible sets of three colors (out of six) we are left with only eight possibilities: A–B–C, A–B–3, A–2–C, A–2–3, 1–B–C, 1–B–3, 1–2–C, and 1–2–3.

(3) Since it is also true that, on any other piece, colors that oppose each other cannot appear in the same solution, we may choose another piece, say a flat, and use it to reduce further the possible solution colors. For instance, it may happen that on that flat, 2 opposes A. We would then know to eliminate anything with A and 2. This would force A and B to be together. One or two more tests with other pieces lead to A–B–C (or 1–2–3) as a candidate for a solution. When eliminating possibilities, it is convenient to turn the three flats to allowed colors, using Biggie as a guide.

(4) Place Biggie as in the diagram so that A–B–C is a corner. Place all three flats so that A, B, and C are showing. These three flats must cover all or part of the colors 1, 2, and 3 of Biggie. Of course, keep A opposite A, etc. It is easy to visualize exactly

where a particular flat must go by looking at its A–B–C corner. Then place the longs and, finally, Smally.

The solver will notice that the completed cube has a "fault-free" property. That is, no seam runs completely through the cube in any direction and therefore no rotation of any part of the cube is possible. This will assure that only one solution is attainable with the colors A–B–C (the proof is left to the reader). There is another solution to this puzzle using the colors 1–2–3. If only one solution is desired, take any piece and interchange two of the colors A, B, and C (or two of 1, 2, and 3). This will change the parity on one corner of the cube so that a solution is impossible using that set of three colors.

We like to have three Smallys prepared: one that yields two solutions, one that gives only one solution, and one to slip in when our enemies try the puzzle that is colored so that *no* solution is possible!

Magic Die

Figure 3 shows the schematic for a Magic Die. The Magic Die has the amazing property that the sum of any row, column, main diagonal (upper left to lower right), or off-diagonal around all four lateral faces is always 42.

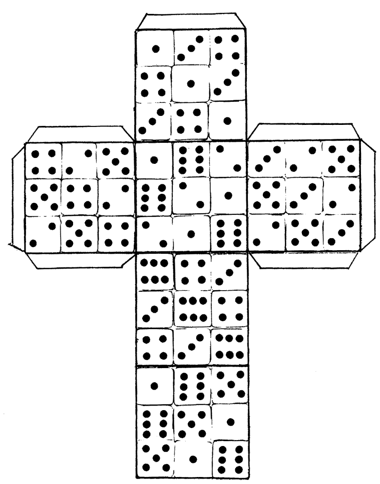

Figure 3.

You can construct a Magic Die puzzle by taking twenty-seven dice and gluing them together into the eight pieces of our combinatorial cube puzzle—being sure the dice conform to the layout shown in Figure 3. There will be only one solution to this puzzle (with magic constant 42), and, even with the schematic as a guide, it will be extremely difficult to find. (Alternatively, you can copy Figure 3 and paste it onto heavy paper to make a permanent Magic Die.)