Point Mirror Reflection

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The Problem

It is well known that a ray of light can reflect many times between two ordinary line mirrors. If we introduce the condition that a ray should reflect only one point on the mirror — reducing the mirrors to point mirrors — we find a maximum of three reflections for two mirrors and seven reflections for three mirrors, if these are suitably placed and oriented. Solutions are shown in Figure 1.

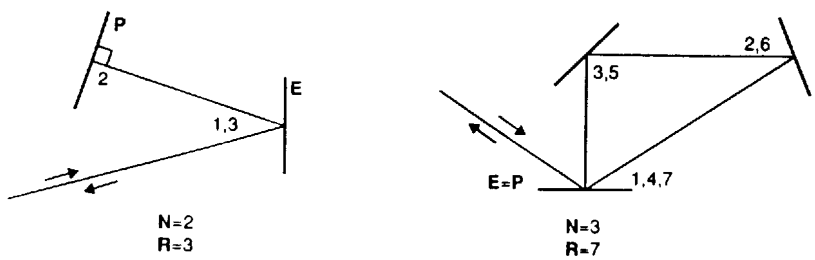

Figure 1. Maximum number of reflections for two and three point mirrors (E = entrance mirror, P = perpendicularly reflecting mirror).

This suggests an intriguing problem: How must N point mirrors be placed and oriented to reflect an entering ray of light as often as possible between these mirrors? What is the maximum number of reflections for varying N?

Upper Limits. From Figure 1 we can see that after some reflections the ray of light is reflected perpendicularly onto a mirror P and returns along the same paths but in the opposite direction. If the ray would not return along the entry paths, passing once again by entrance mirror E, half of the potential reflections would not be used. It is obvious that the strategy of

Figure 2. Number of reflections for different mirror types.

using a perpendicularly reflecting mirror P yields the maximum number of reflections.

At each mirror a maximum of $N-1$ rays can be reflected, going to every other mirror. At mirror E there can be one additional reflection for the entering ray. Therefore, the theoretical maximum number of reflections is $N(N-1)+1$, or

$$R_1 = N^2 - N + 1\,. \tag{1}$$

However, for even N there is a parity complication. Only at mirror P can there be an odd number of reflections. At all other mirrors there must be an even number of reflections, since the light paths are used in two directions; see Figure 2.

At mirror E there can be at most N reflections, $N-1$ from the other mirrors and one from the entering ray. At mirror P there can be at most $N-1$ reflections, from the other mirrors, which is an odd number. However, at the other $N-2$ mirrors (other than the E and P mirrors) there can't be $N-1$ reflections, but only $N-2$ to make it an even number. So the upper limit for an even number of mirrors is $N+(N-1)+(N-2)(N-2)$ or

$$R_2 = N^2 - 2N + 3\,. \tag{2}$$

For odd N the parities are okay if E and P are the same mirror, so R_1 remains valid.

Mirrors on the Corners of a Regular Polygon

Can the theoretical maxima R_1 and R_2 be achieved for any N, and what should the positions and the orientations of the mirrors be? A well-known geometrical property may help us: When the N mirrors are put on the corners of a regular N-gon, the angle between all the neighboring pairs of light paths is $180°/N$, as shown for the regular pentagon in Figure 3.

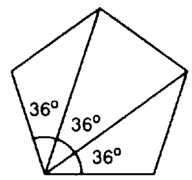

Figure 3. Light paths in a regular pentagon.

Using such an N-gon grid, two-dimensional space is sufficient to find solutions for each N. Only discrete orientations of the mirror are allowed and are multiples of $90°/N$, say $m \times 90°/N$. For $m = 0$ we have the normal orientation, $m = 1$ yields a rotation of one step clockwise, and so on. This is illustrated in Figure 4.

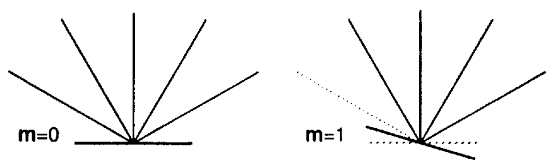

Figure 4. Rotation of a mirror.

Prime Number of Mirrors. For prime N the maximum number of reflections can always be achieved. We take $m = 1$ for E (= P) and $m = 0$ for all other mirrors. The ray of light, starting at mirror E, makes hops of one mirror; after passing mirror E again, it makes hops of two mirrors, and so on. Since N is prime, the ray will pass every mirror once at each tour from E to E; see Figure 5.

Non-Prime Number of Mirrors. If the number of mirrors is non-prime, the investigation of various cases is less straightforward. A systematical trial-and-error search for the maximum number of reflections may be performed using the following strategy:

Step 1. Draw the N-gon and all possible light paths (the N-gon grid). Add the starting ray at an angle of $180°/N$ with the N-gon.

Step 2. Omit one or more light paths (lines of the N-gon grid), and adjust the mirror orientations such that symmetry of light paths is preserved at every mirror and such that only at mirror P is there a perpendicular reflection. Prevent any obvious loops, since a ray can neither enter nor leave such a loop.

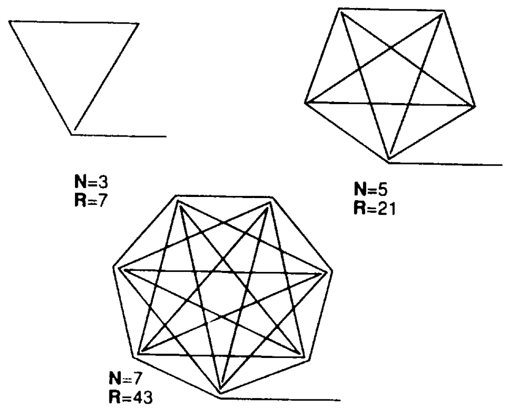

Figure 5. Reflection diagrams for prime numbers of mirrors.

Step 3. Count the number of reflections by following the entering ray. Calculate whether the number of reflections is equal to two times the number of remaining light paths plus one. If this is true, there are no loops and all of the remaining light paths are used. If not, reject the candidate solution.

Repeat Steps 2 and 3 for all candidates missing one light path, then for all candidates missing two light paths, and so on, until one or more solutions are found.

Figure 6 illustrates the procedure for the case of six mirrors. At least two light paths have to be omitted for Step 2. The candidate has 13 light paths. The grid consists of 15 lines, a choice of a set of 2 points out of 6, decreased by 2 to account for the loss of the two vertical lines on the left and right sides. In this case, however, we do not count $2 \times 13 + 1 = 27$ reflections but only 19. Indeed, there is a loop and the candidate has to be rejected.

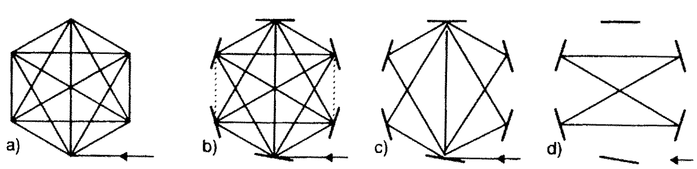

Figure 6. Searching strategy for $N = 6$; (a) Hexagon grid plus entering ray; (b) A candidate having 19 reflections and a loop, shown separately in (c) and (d).

N	R	d	m															
4	11	1	0	*0	0													
6	27	2	1	1	-1	1	-1	*0										
6	27	2	1	0	1	-1	0	*0										
8	51	3	1	1	-1	*0	1	-1	1	-1								
9	69	2	1	0	0	0	*1	-1	0	0	0							
10	83	4	1	1	-1	*0	0	1	-1	1	-1	0						
12	123	5	1	1	-1	1	-1	*0	1	-1	1	-1	1	-1				
14	171	6	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	*0		
15	199	6	*1	0	0	0	0	0	0	0	0	2	0	-2	1	-1	0	
16	227	7	1	1	-1	*0	0	0	0	0	1	-1	1	-1	0	0	0	0
		$c \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

N = number of mirrors. c = corner number. m = mirror orientation number.

R = number of reflections. d = number of deleted paths.

The entrance mirror E is positioned in corner number one ($c_E = 1$).

The position of the perpendicular reflecting mirror P is marked with an asterisk*.

Figure 7. Data for composite N up to 16.

Some results obtained by applying the strategy outlined above are summarized in Figures 7 and 8.

- For N composite and odd the theoretical maximum of R_1 can never be achieved. If no light paths are omitted, m must be 0 for all mirrors, except for E, which has $m = 1$. There will always be an N/n loop, in which n stands for any divisor of N. For this case a sharper theoretical maximum might be defined, since light paths have to be omitted to account for all N/n loops.
- For even N this sharper maximum is given by R_2. For $N = 2$ to $N = 16$, the value R_2 can be achieved. Actual solutions for $N = 18$ or $N = 20$ have not been found, but might be possible.
- For $N = 6$ there are two solutions; see Figures 7, 8, and 9. For all other N up to 16 there is only one solution.

A general solution of the problem has not been found, nor a proof that a solution can always be put on our N-grid. It is likely that this is true, since only the N-gon grid seems to have a geometry that suits the problem.

Conclusion

A problem was defined regarding the maximum number of reflections of a ray of light between N point mirrors. A theoretical maximum was found,

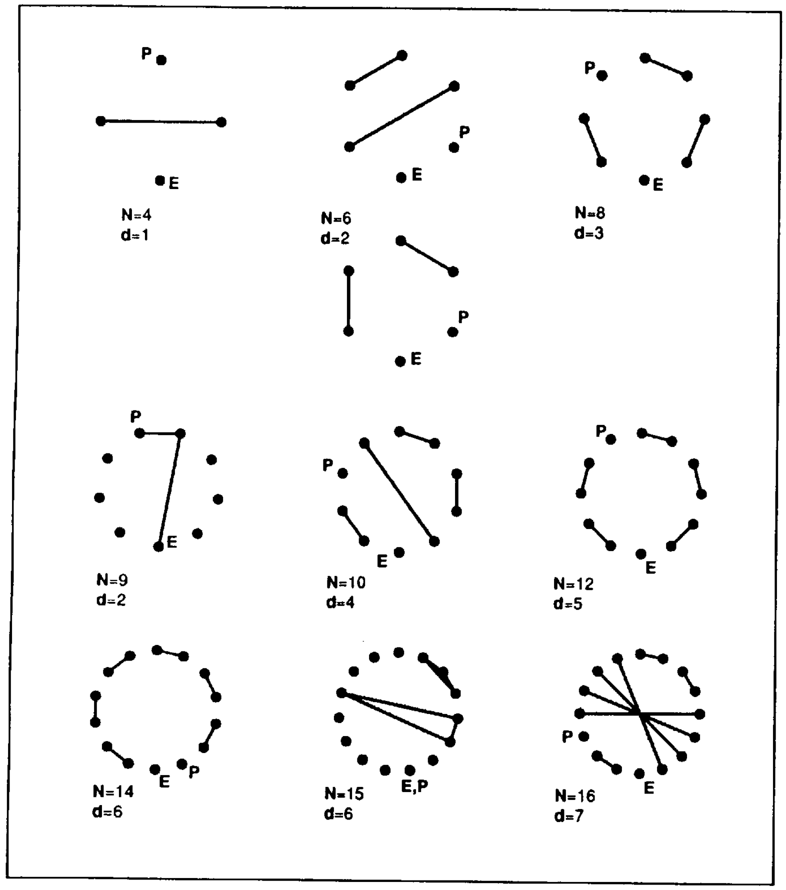

Figure 8. Deleted paths for composite N up to 16.

which can be reached for prime N. For composite N up to 16, solutions were found by a systematic trial and error search using a regular N-gon grid. No general strategy was found, and many questions remain open:

- Is there a better search strategy for composite N?
- What are the properties of solutions for $N = 18$ and higher?
- Are there more N with multiple solutions?
- Can a sharper theoretical maximum for odd and composite N be expressed in a closed form?
- Are there solutions that use the back side of the mirrors?

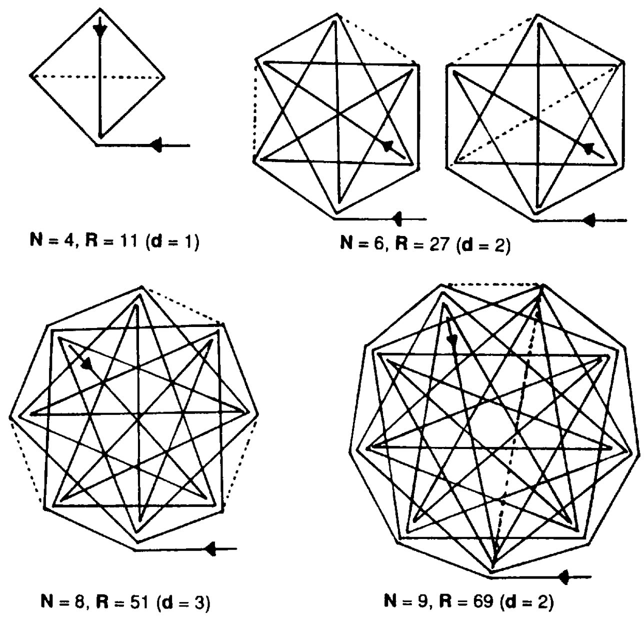

Figure 9. Reflection diagrams for composite N up to 9.

- Are there solutions that can't be put on an N-gon grid or that require three-dimensional space?
- Is there a general theory to the problem?
- Are there applications to, for example, lasers or burglar alarms?