Block-Packing Jambalaya

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My primary interest over the years has been burr puzzles, but there is another small category of puzzles that is especially intriguing to me. It is 3-dimensional box-packing puzzles where the box and all the pieces are rectangular solids. The number of such puzzles that I am aware of is quite small, but the "tricks," or unique features that the puzzles employ are many and varied. I know of no other small group of puzzles that encompasses such a rich diversity of ideas.

Presented here are 11 such "block-packing" puzzles. The tricks to most of the puzzles are discussed here, but complete solutions are not given.

The puzzles are grouped according to whether there are holes in the assembled puzzle and whether the pieces are all the same or different.

For each puzzle, the total number of pieces is in parentheses. If known, the inventor of the puzzle, date of design, and manufacturer are given.

1. No Holes, All Pieces the Same

"Aren't these puzzles trivial?," you ask. Well, you are not far from being completely correct, but there are some interesting problems. David Klarner gives a thorough discussion of this case in [5]. The following are my favorites:

1. unnamed (44) (Singmaster–Klarner):
   Box: $8 \times 11 \times 21$
   Pieces: (44) $2 \times 3 \times 7$
2. unnamed (45) (de Bruijn):
   Box: $5 \times 6 \times 6$
   Pieces: (45) $1 \times 1 \times 4$

2. No Holes, Limited Number of Piece Types

The puzzles I know of in this category follow a common principal: There are basically two types of pieces — a large supply of one type and a limited supply of another. The pieces of the second type are smaller and easier to use, but must be used efficiently to solve the puzzle. The solver must determine exactly where the second set of pieces must be placed, and then the rest is easy.

3. unnamed (9) (Slothouber–Graatsma):
   Box: $3 \times 3 \times 3$
   Pieces: (3) $1 \times 1 \times 1$, (6) $1 \times 2 \times 2$
4. unnamed (18) (John Conway):
   Box: $5 \times 5 \times 5$
   Pieces: (3) $1 \times 1 \times 3$, (1) $1 \times 2 \times 2$, (1) $2 \times 2 \times 2$, (13) $1 \times 2 \times 4$

In the first of these, the three individual cubes are obviously easy to place, but they must not be wasted. By analyzing "checkerboard" colorings of the layers in the box, it is easy to see that the cubes must be placed on a main diagonal. In the second design, the three $1 \times 1 \times 3$ pieces must be used sparingly. The rest of the pieces, although not exactly alike, function similarly to the $1 \times 2 \times 2$ pieces in the first puzzle. See [2] or [5] for more information.

3. No Holes, Pieces Mostly Different

5. Quadron (18) (Jost Hanny, Naef):
   Box 1: $5 \times 7 \times 8$
   Pieces: $2 \times 3 \times 3$, $2 \times 3 \times 5$, $2 \times 4 \times 5$, $2 \times 4 \times 6$, $3 \times 3 \times 4$, $3 \times 3 \times 5$, $3 \times 3 \times 7$
   Box 2: $5 \times 7 \times 10$
   Pieces: $1 \times 3 \times 4$, $1 \times 3 \times 6$, $1 \times 3 \times 7$, $1 \times 3 \times 10$, $1 \times 4 \times 5$, $2 \times 3 \times 4$, $2 \times 3 \times 6$, $2 \times 3 \times 7$, $2 \times 4 \times 7$, $3 \times 3 \times 3$, $4 \times 4 \times 4$
   Box 3: $7 \times 9 \times 10$
   Pieces: all 18 pieces from first two boxes

Quadron does not use any special tricks that I am aware of, but it does make a nice set of puzzles. The 18 pieces are all different, and the three boxes offer a wide range of difficulty. The small box is very easy. The seven pieces can be placed in the box in ten different ways, not counting rotations and/or reflections. A complete, rigorous analysis of the puzzle can be done by hand in about 15 minutes. The middle-size box is difficult —

there is only one solution. The large box is moderately difficult, and has many solutions.

Quadron also makes for a nice entrance into the realm of computer analysis of puzzles and the limitations of such programs. The programmer can use algorithms that are used for pentominoe problems, but there are more efficient algorithms that can be used for block-packing puzzles. I wrote such a program on my first computer, a Commodore 64. The program displayed the status of the box at any instant using color graphics. I painted pieces of an actual model to match the display. The result was a fascinating demonstration of how a computer can be used to solve such a puzzle. The Commodore 64 is such a wonderously slow machine — when running the program in interpreter BASIC, about once a second a piece is added or removed from the box! Using compiled BASIC, the rate increases to 40 pieces/second.

When running these programs on more powerful computers, the difference between the three boxes is stunning: The first box can be completely analyzed in a small fraction of a second. The second box was analyzed in about a minute of mainframe computer time. In early 1996, I did a complete analysis of the third box. There are 3,450,480 solutions, not counting rotations and reflections. The analysis was done on about 20 powerful IBM workstations. The total CPU time used was about 8500 hours, or the equivalent of one year on one machine. By the end of the runs, the machines had constructed 2 1/2 trillion different partially filled boxes.

6. Parcel Post Puzzle (18) (designer unknown; copied from a model in the collection of Abel Garcia):

   Box: $6 \times 18 \times 28$

   Pieces: all pieces are of thickness 2 units; the widths and lengths are

   4×9, 5×18, 5×21, 6×7, 6×10, 6×13, 7×18, 8×18, 9×11, 9×13, 10×11, 11×11 and two each of 5×9, 7×8, and 7×13.

Since all the pieces are of the same thickness and the box depth equals three thicknesses, it is tempting to solve the puzzle by constructing three layers of pieces. One or two individual layers can be constructed, but the process cannot be completed. The solution involves use of the following obvious trick (is that an oxymoron?): Some piece(s) are placed sideways in the box. Of the 18 pieces, 10 are too wide to fit into the box sideways and 4 are of width 5, which is no good for this purpose. This leaves 4 pieces that might be placed sideways. There are four solutions to the puzzle, all very similar, and they all have three of these four pieces placed sideways.

7. Boxed Box (23) (Cutler, 1978, Bill Cutler Puzzles):

   Box: $147 \times 157 \times 175$

Pieces: $13\times112\times141$, $14\times70\times75$, $15\times44\times50$, $16\times74\times140$, $17\times24\times67$, $18\times72\times82$, $19\times53\times86$, $20\times40\times92$, $21\times52\times65$, $22\times107\times131$, $23\times41\times73$, $26\times49\times56$, $27\times36\times48$, $28\times55\times123$, $30\times54\times134$, $31\times69\times78$, $33\times46\times60$, $34\times110\times135$, $35\times62\times127$, $37\times83\times121$, $38\times42\times90$, $45\times68\times85$, $57\times87\times97$

The dimensions of the pieces are all different numbers. The pieces fit into the box with no extra space. The smallest number for which this can be done is 23. There are many other 23-piece solutions that are combinatorially different from the above design. Almost 15 years later, this puzzle still fascinates me. See [1] or [3] for more information.

4. Holes, Pieces the Same or Similar

8. Hoffman's Blocks (27) (Dean Hoffman, 1976)
   Box: $15\times15\times15$
   Pieces: (27) $4\times5\times6$

This sounds like a simple puzzle, but it is not. The extra space makes available a whole new realm of possibilities. There are 21 solutions, none having any symmetry or pattern. The dimensions of the pieces can be modified. They can be any three different numbers, where the smallest is greater than one-quarter of the sum. The box is a cube with side equal to the sum. I like the dimensions above because it tempts the solver to stack the pieces three deep in the middle dimension. See [4].

9. Hoffman Junior (8) (NOB Yoshigahara, 1986, Hikimi Puzzland)
   Box: $19\times19\times19$
   Pieces: Two each of $8\times9\times10$, $8\times9\times11$, $8\times10\times11$, $9\times10\times11$

5. Holes, Pieces Different

10. Cutler's Dilemma, Simplified (15) (Cutler, 1981, Bill Cutler Puzzles)
    Box: $40\times42\times42$
    Pieces: $9\times19\times26$, $9\times20\times20$, $10\times11\times42$, $10\times12\times26$, $10\times16\times31$, $10\times19\times25$, $10\times19\times26$, $11\times11\times25$, $11\times12\times42$, $11\times16\times19$, $11\times17\times30$, $11\times19\times25$, $12\times17\times19$, $16\times19\times21$, $17\times19\times21$

The original design of Cutler's Dilemma had 23 pieces and was constructed from the above, basic, version by cutting some of the pieces into two or three smaller pieces. The net result was a puzzle that is extremely difficult. I will not say anything more about this design except that the trick involved is different from any of those used by the other designs in this paper.

6. Miscellaneous

11. Melting Block (8-9) (Tom O'Beirne)
    Box: $58 \times 88 \times 133$
    Pieces: $19 \times 29 \times 44$, $19 \times 29 \times 88$, $19 \times 58 \times 44$, $38 \times 29 \times 44$, $19 \times 58 \times 88$, $38 \times 29 \times 88$, $38 \times 58 \times 44$, $38 \times 58 \times 88$
    plus a second copy of $19 \times 29 \times 44$

The Melting Block is more of a paradox than a puzzle. The eight pieces fit together easily to form a rectangular block $57 \times 87 \times 132$. This fits into the box with a little room all around, but seems to the casual observer to fill up the box completely. When the ninth piece is added to the group, the pieces can be rearranged to make a $58 \times 88 \times 133$ rectangular solid. (This second construction is a little more difficult.) This is a great puzzle to show to "non-puzzle people" and is one of my favorites.

By the way, one of the puzzles listed above is impossible. I won't say which one (it should be easy to figure out). It is a valuable weapon in every puzzle collector's arsenal. Pack all the pieces, except one, into the box, being sure that the unfilled space is concealed at the bottom and is stable. Place the box on your puzzle shelf with the remaining piece hidden behind the box. You are now prepared for your next encounter with a boring puzzle-nut. (No, readers, this is not another oxymoron, but rather a tautology to the 99% of the world that would never even have started to read this article.) Pick up the box and last piece with both hands, being careful to keep the renegade piece hidden from view. Show off the solved box to your victim, and then dump the pieces onto the floor, including the one in your hand. This should keep him busy for quite some time!

References

[1] W. Cutler, "Subdividing a Box into Completely Incongruent Boxes," *J. Recreational Math.,* 12(2), 1979–80, pp. 104–111.
[2] M. Gardner, Mathematical Games column in *Scientific American,* February 1976, pp. 122–127.
[3] M. Gardner, Mathematical Games column in *Scientific American,* February 1979, pp. 20–23.

[4] D. Hoffman, "Packing Problems and Inequalities," in *The Mathematical Gardner,* edited by D. Klarner (Wadsworth International, 1981), pp. 212–225.
[5] D. Klarner, "Brick-Packing Puzzles," *J. Recreational Math.,* 6(2), Spring 1973, pp. 112–117.

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