Polly's Flagstones

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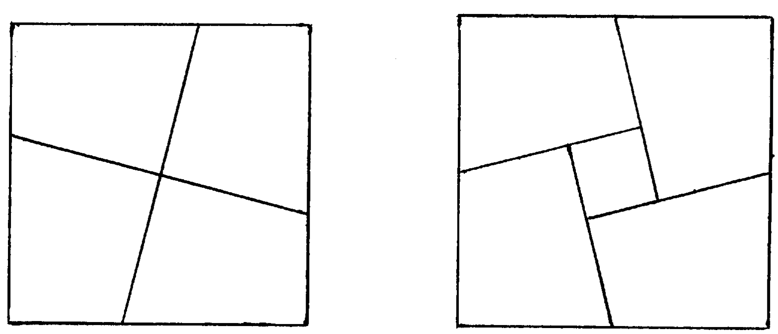
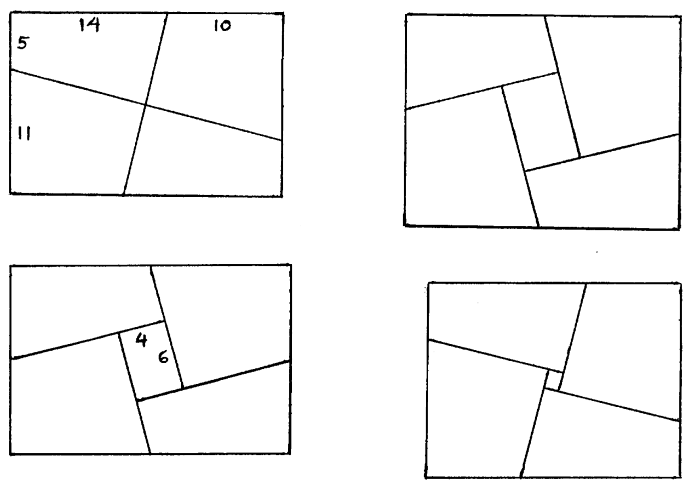
I wish to report on some recent correspondence with my good friends the Gahns in Calcutta. You may remember meeting Paul in *The Puzzling World of Polyhedral Dissections*. His wife Polly is an avid gardener. She presented me with the following problem.

Polly places precisely fitted flagstones around various plantings in her garden in order to suppress weeds. She and Paul have a bent for geometrical recreations, so they are always on the lookout for creative and original solutions to their various landscaping projects. She had one large stone that was perfectly square. She asked me if it were possible to cut the stone into four pieces that could be arranged to form a somewhat larger square perimeter enclosing a square hole having sides one quarter those of the original stone. The simple solution to this problem which I then sent to her, a classic dissection of the square, is shown below.



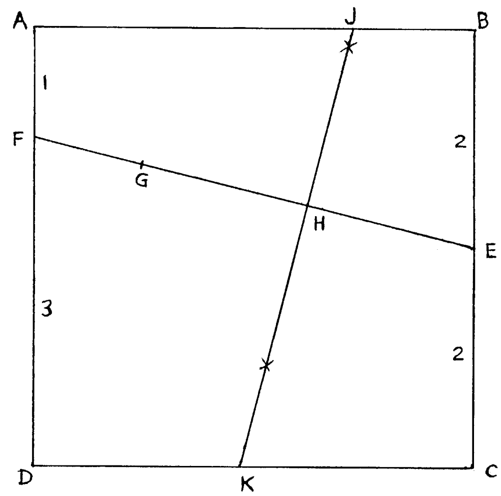
In order to meet Polly's required dimensions, the sides are divided in the ratio of 3 to 5. Frankly, I was surprised that Polly would bother to write about such a trivial dissections problem, but I should have known better. She wrote back with profuse thanks for such an "elegant" solution, and her sarcasm rather put me on guard.

Now she posed another problem. She decided that she preferred mostly rectangular rather than square planting spaces. Suppose we were to consider starting with a large rectangular stone, which she proposed that we cut into four pieces, but this time to form a rectangular perimeter enclosing a rectangular planting space. And for good measure, to allow for more flexibility in landscaping, how about a scheme whereby the stones could be rearranged to form any one of three different-sized rectangular openings, each one enclosed by a rectangular perimeter. This required a bit more reflection than the first problem, but after tinkering for a while with paper, pencil, and scissors, I came up with the scheme shown below.



Almost any rectangle dissected symmetrically by two mutually perpendicular lines that touch all four sides will produce four quadrilaterals that can be rearranged to create the required three different rectangular holes enclosed by rectangles. One of these rectangular holes will have the same shape as the original rectangle and will be surrounded by a rectangle also having that same shape. With the dimensions shown, the medium-sized rectangular hole will have dimensions exactly one quarter those of the original stone. The smaller and larger holes have dimensions that are irrational.

If you think that Polly was satisfied with this solution, then you don't know Polly very well. She immediately wrote back and suggested that it was a shame to cut up two beautiful large stones when one might do, cut into four pieces, which could then be rearranged to create a square hole within a square enclosure or any one of three different rectangular holes within rectangular enclosures. It was then I realized that from the start she had just been setting me up. I decided to play into it, so I wrote back and asked what made her so sure it was even possible. Sure enough, by return mail arrived her solution, shown below.



For the purpose of this example, let the original square stone *ABCD* be four feet on a side. Locate the midpoint *E* of side *BC*. The location of point *F* is arbitrary, but for this example let it be one foot from *A*. Draw *EF*. Subtract length *CE* from length *DF*, and use that distance from *F* to locate point *G*. Bisect *EG* to locate *H*, and draw *JHK* perpendicular to *EHF* (most easily done by swinging arcs from *E* and *G*).

Don't ask my why this works, but it does. The actual arrangements of the pieces for the four different solutions, one square and three rectangular, are left for the reader to discover. One of the rectangles has a pair of solutions — the others are unique. With these dimensions, the square hole will be one foot square. For added recreation, note that the pieces can also be arranged to form a solid parallelogram (two solutions), a solid trapezoid (two solutions), and a different trapezoid (one solution).

An interesting variation is to let points *A* and *F* coincide, making one of the pieces triangular. The same solutions are possible, except that one of the trapezoids becomes a triangle.

Now what sort of scheme do you suppose Polly will come up with for that other stone, the rectangular one?